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Self-intersection local time for Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes: Existence, path continuity and examples

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Abstract

In this paper we develop a criterion for existence or non-existence of self-intersection local time (SILT) for a wide class of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-valued processes, we show that quite generally the SILT process has continuous paths, and we give several examples which illustrate existence of SILT for different ranges of dimensions (e.g., $d \leqslant 3, d \leqslant 7$ and $5 \leqslant d \leqslant 11$ in the Brownian case). Some of the examples involve branching and exhibit "dimension gaps". Our results generalize the work of Adler and coauthors, who studied the special case of "density processes" and proved that SILT paths are cadlag in the Brownian case making use of a "particle picture" approximation (this technique is not available for our general formulation).

Keywords: Self-intersection local time; Gaussian $\mathscr{S}'(\mathbb{R}^d)$-valued process; Density process, Branching; Dimension gap.

1. Introduction

This paper was inspired by our reading of the works of Adler and coauthors (Adler, 1993, Adler, et al., 1991; Adler and Lewin, 1991,1992; Adler and Rosen, 1993), to whom we shall sometimes refer collectively as $\mathbb{A}$, on self-intersection local time (SILT) of the so-called "density processes".

The SILT of a process $X = (X_t)_{t\in[0,T]}$ taking values in the space $\mathscr{S}'(\mathbb{R}^d)$ (the tempered distributions on $\mathbb{R}^d$), up to time t, is defined intuitively by the formal expression

$$\int_0^t \int_0^t \langle X_s \otimes X_r, \delta(x-y)\varphi(y)\rangle \mathrm{d}s\, \mathrm{d}r, \tag{1.1}$$

where $\otimes$ denotes the tensor product in $\mathscr{S}'(\mathbb{R}^d), \delta$ is the Dirac distribution, $\varphi \in \mathscr{S}(\mathbb{R}^d)$ (the rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$), and $\langle\ ,\ \rangle$ designates

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the duality on $\mathscr{S}'(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$. Obviously, this expression does not make mathematical sense, and the question is how to give a rigorous meaning to it, at least for some interesting $\mathscr{S}'(\mathbb{R}^d)$-processes, for example, some Gaussian processes which arise in applications. $\mathbb{A}$ have achieved this for Brownian and for α-stable (non-Brownian) density processes. These processes are high-density fluctuation limits of certain particle systems. Since the density processes are very special and simple but the intuitive definition of SILT is quite general, we were interested in investigating if the definition can be given a rigorous meaning for a much wider class of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes, and what it tells about specific examples which are not as simple as the density processes and which are also of interest in applications. We were also intrigued by the lack of a result on continuity of SILT paths, since SILT is defined as a limit of "approximate" SILTs, which are continuous.

Given our purpose, we decided to restrict the present study to the basic SILT (i.e., of order 2) and to the question of existence, non-existence and continuity of SILT. $\mathbb{A}$ study also SILT of higher orders, divergence limits and Tanaka-type formulas for density processes, but these kinds of results are not directly relevant to our objectives in this paper.

Our contributions, on the background of $\mathbb{A}$, are the following.

(1) We give a general scheme, which we formulate as a theorem (Theorem 2.4), for verifying existence of SILT for any given centered Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process, and showing that (under a simple additional condition which is met in applications) the SILT process has continuous paths. $\mathbb{A}$ prove that SILT for the Brownian density process has cadlag paths (Theorem 3.2 in Adler et al., 1991). Hence, our result is a generalization in the following ways: it encompasses a wide class of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes, SILT paths are continuous, and in the case of density processes it includes also the stable (non-Brownian) case. We stress that Theorem 2.4 is a scheme for proving existence and continuity of SILT for given Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes, but since the scheme is quite general, its application to specific cases may be difficult. Lemma 2.6 gives sufficient conditions for existence and continuity of SILT which are satisfied in several applications, and Corollary 2.5 gives a sufficient condition for non-existence of SILT.

(2) $\mathbb{A}$ discuss an interpretation of SILT for density processes in terms of an approximating particle system, the so-called "particle picture", and they make use of this approximation to prove the cadlag property of SILT paths in the Brownian case (Theorem 3.2 in Adler, et al., 1991). In a note at the bottom of p. 201 in Adler, et al. (1991) it is stated that the cadlag property is not available in the existence proof of SILT as an L^2-limit. However, we found that it is possible to exploit the L^2-limit existence proof to show SILT path continuity. Although for some models the particle picture constitutes an important approach which possesses intuitive content, we think it is desirable, for mathematical generality and self-consistency, to derive SILT properties from the definition alone. This is further justified by the fact there are Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes that possess SILT but for which there is no naturally associated particle picture.

(3) $\mathbb{A}$ prove that SILT for the Brownian density process on $\mathbb{R}^d$ exists for dimensions $d \leqslant 3$ ($d < 2\alpha$ for the symmetric α-stable case). We give examples of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes which illustrate existence of SILT for different ranges of dimensions

(e.g., $d \leqslant 3, d \leqslant 7$ and $5 \leqslant d \leqslant 11$ in the Brownian case, the results being given for the symmetric α-stable case). The example with $5 \leqslant d \leqslant 11$ involves superprocesses. We also give examples where SILT does not exist for any dimension. Most of the previous examples arise from fluctuation limits of particle systems. We consider also (inhomogeneous) Wiener $\mathscr{S}'(\mathbb{R}^d)$-processes. These processes do not necessarily arise from fluctuations of particle systems, and some of them possess SILT but some do not. Wiener $\mathscr{S}'(\mathbb{R}^d)$-processes also serve to illustrate the fact that the definition of SILT for $\mathscr{S}'(\mathbb{R}^d)$-processes is not an extension of the definition for the finite-dimensional case. In the examples involving branching there may or may not be a "dimension gap" caused by the clumping effect of the branching, depending on the type of the underlying scaling limit. These results help to further understand this phenomenon, which is discussed in Adler (1993) in connection with the relationship between SILTs for density processes and superprocesses.

(4) Concerning the techniques for the analysis of $\mathscr{S}'$-processes, we have endeavored to give self-contained proofs and to make them accessible to probabilists who possess elementary knowledge of $\mathscr{S}'$-processes and functional analysis. We stress the important role played by the nuclear Fréchet property of the space $\mathscr{S}(\mathbb{R}^d)$ and we give a simple jusfitication for the analogue of the Wick product used in the definition of SILT. We felt the need to do this because we are not familiar with some of the techniques used by $\mathbb{A}$ and we could not find sources in the literature that we could consider sufficiently rigorous, in particular on the "standard fare" mentioned by them in connection with Wick products and Gaussian $\mathscr{S}'$-distributions, with a reference to Glimm and Jaffe (1981).

Section 2 contains the general results on existence, non-existence and continuity of SILT. In Section 3 we give the examples, including Wiener and Ornstein–Uhlenbeck $\mathscr{S}'$-processes, and also processes that are not Markovian (among the latter are occupation time processes). Section 4 is devoted to a brief discussion of dimension gaps in the examples involving branching; but since this problem is not an objective of this paper, we only make some general comments and raise some questions which require further thought. The proofs for Sections 2 and 3 are collected in Sections 5 and 6, respectively.

Basic background on the space $\mathscr{S}'(\mathbb{R}^d)$ and on $\mathscr{S}'(\mathbb{R}^d)$-valued random variables and stochastic processes can be found, e.g., in Treves (1967) and Itô (1984), respectively.

2. General definitions and results: existence and continuity of SILT

We will consider $\mathscr{S}'(\mathbb{R}^d)$-valued processes with time interval $[0,1]$.

The following preparatory result will be used to give a rigorous meaning to the formal expression (1.1).

Proposition 2.1. *Let $X = (X_t)_{t\in[0,1]}$ be a continuous centered Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process. Then there exists a unique (up to modification) measurable random field*

$$(: X_s \otimes X_t :)_{(s,t)\in[0,1]\times[0,1]}$$

in $\mathscr{S}'(\mathbb{R}^{2d})$ *such that*

$$\langle :X_s \otimes X_t:, \varphi \otimes \psi \rangle = \langle X_s, \varphi \rangle \langle X_t, \psi \rangle - E(\langle X_s, \varphi \rangle \langle X_t, \psi \rangle) \tag{2.1}$$

for each $\varphi, \psi \in \mathscr{S}(\mathbb{R}^d)$, $s, t \in [0,1]$. *Moreover, the mapping* $\Phi \mapsto \langle :X_. \times X_.:, \Phi \rangle$ *is continuous linear as a function from* $\mathscr{S}(\mathbb{R}^{2d})$ *into* $L^2([0,1] \times [0,1] \times \Omega)$.

Throughout the paper $\langle\ ,\ \rangle$ always denotes the duality on the appropriate spaces. The notation $:X_s \otimes X_t:$ is chosen to stress an analogy to the "physical Wick product" (see Gjessing et al., 1993).

We will need, as in $\mathbb{A}$, the following class of approximating functions of the Dirac δ.

Notation. Let $\mathscr{F}$ = the class of non-negative symmetric functions $f \in C_0^\infty(\mathbb{R}^d)$ (compact support) such that $\int_{\mathbb{R}^d} f(x)dx = 1$. For $f \in \mathscr{F}$, $\varepsilon > 0$, let

$$f_\varepsilon(x) = \varepsilon^{-d} f\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d.$$

For $f \in \mathscr{F}$, $\varepsilon > 0$, $\varphi \in S(\mathbb{R}^d)$, let

$$\Phi^f_{\varepsilon,\varphi}(x,y) = \varphi(x) f_\varepsilon(x-y), \quad x, y \in \mathbb{R}^d. \tag{2.2}$$

The mapping $\varphi \mapsto \Phi^f_{\varepsilon,\varphi}$ is continuous linear from $\mathscr{S}(\mathbb{R}^d)$ into $\mathscr{S}(\mathbb{R}^{2d})$. Hence, by Proposition 2.1, $\varphi \mapsto \langle :X_. \otimes X_.:, \Phi^f_{\varepsilon,\varphi} \rangle$ is a continuous linear mapping from $\mathscr{S}(\mathbb{R}^d)$ into $L^2([0,1] \times [0,1] \times \Omega)$. This and the regularization theorem (e.g., Itô, 1984) imply that the formula

$$\langle L^f_\varepsilon(t), \varphi \rangle = \int_0^t \int_0^t \langle :X_s \otimes X_r:, \Phi^f_{\varepsilon,\varphi} \rangle \, \mathrm{d}s \, \mathrm{d}r, \quad t \in [0,1], \ \varphi \in \mathscr{S}(\mathbb{R}^d) \tag{2.3}$$

defines an $\mathscr{S}'(\mathbb{R}^d)$-valued process L^f_ε. Moreover, for any fixed $\varphi \in \mathscr{S}(\mathbb{R}^d)$ the real process $\langle L^f_\varepsilon, \varphi \rangle$ is clearly continuous, and therefore by the Mitoma criterion (Mitoma, 1983) we have obtained the following result.

Corollary 2.2. *For each* $\varepsilon > 0, L^f_\varepsilon$ *defined by formula* (2.3) *is a continuous* $\mathscr{S}'(\mathbb{R}^d)$-*process.*

$L^f_\varepsilon(t)$ is regarded, for small ε, as an approximation for the intuitive expression (1.1). Note, however, the centering or normalizing term in formula (2.1). The need for a normalization in order to obtain convergence when letting $\varepsilon \to 0$ is now a standard fact in the theory of (self-) intersection local times (cf. $\mathbb{A}$).

Definition 2.3. If there exists an $\mathscr{S}'(\mathbb{R}^d)$-process $L = (L(t))_{t \in [0,1]}$ such that for each $t \in [0,1], \varphi \in \mathscr{S}(\mathbb{R}^d)$ and any $f \in \mathscr{F}, \langle L(t), \varphi \rangle$ is the mean-square limit of $\langle L^f_\varepsilon(t), \varphi \rangle$ as $\varepsilon \to 0$, then the process L is called the self-intersection local time (SILT) of the process X.

We will now describe a general scheme for proving existence and continuity of SILT.

Let X be a process as in Proposition 2.1, and denote by $K(s,\varphi;t,\psi)$ the covariance functional of X, i.e.,

$$K(s,\varphi;t,\psi) = E(\langle X_s, \varphi \rangle \langle X_t, \psi \rangle), \quad s,t \in [0,1], \ \varphi, \psi \in \mathscr{S}(\mathbb{R}^d).$$

Formula (2.1), together with formulas for fourth moments of Gaussian distributions, imply that for $\Phi, \overline{\Phi} \in \mathscr{S}(\mathbb{R}^d) \otimes \mathscr{S}(\mathbb{R}^d)$ $(\subset \mathscr{S}(\mathbb{R}^{2d}))$ of the form

$$\Phi = \sum_{i=1}^{n} \varphi_i \otimes \psi_i, \qquad \overline{\Phi} = \sum_{j=1}^{m} \overline{\varphi}_j \otimes \overline{\psi}_j, \quad \varphi_i, \psi_i, \overline{\varphi}_j, \overline{\psi}_j \in \mathscr{S}(\mathbb{R}^d), \tag{2.4}$$

we have

$$\begin{aligned} &E(\langle :X_s \otimes X_r :, \Phi\rangle\langle :X_u \otimes X_v :, \overline{\Phi}\rangle) \\ &= \sum_{i,j}(K(s,\varphi_i; u, \overline{\varphi}_j)K(r,\psi_i; v, \overline{\psi}_j) + K(s,\varphi_i; v, \overline{\psi}_j)K(r,\psi_i; u, \overline{\varphi}_j)) \end{aligned} \tag{2.5}$$

for each $s, r, u, v \in [0, 1]$.

The scheme is given in the following result.

Theorem 2.4. *Given a continuous centered Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X, assume that the right-hand side of (2.5) can be written in the form $J_{s,r,u,v}(\Phi, \overline{\Phi})$, the function $J_{s,r,u,v}$ being well-defined on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$ and such that*

(i) *the functional*

$$(\Phi, \overline{\Phi}) \mapsto \int_{[0,t]^4} J_{s,r,u,v}(\Phi, \overline{\Phi}) \, \mathrm{d}s \, \mathrm{d}r \, \mathrm{d}u \, \mathrm{d}v$$

is continuous on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$ for each $t \in [0, 1]$.

(ii) *$J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})$ converges to a finite limit as $\varepsilon, \delta \to 0$ for each $f, \overline{f} \in \mathscr{F}$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $s, r, u, v \in [0, 1]$, and this limit does not depend on $f, \overline{f}$ ($\Phi^f_{\varepsilon,\varphi}$ is given by (2.2)).*

$$\text{(iii)} \quad |J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})| \leqslant G_\varphi(s, r, u, v) \tag{2.6}$$

for some function G_φ on $[0, 1]^4$ which is independent of $\varepsilon, \delta, f, \overline{f}$, and such that

$$\int_{[0,1]^4} G_\varphi(s, r, u, v) \, \mathrm{d}s \, \mathrm{d}r \, \mathrm{d}u \, \mathrm{d}v < \infty \tag{2.7}$$

for each $\varphi \in \mathscr{S}(\mathbb{R}^d)$.

Then the SILT L of the process X exists. Moreover, the mean-square convergence of $\langle L^f_\varepsilon(t), \varphi\rangle$ to $\langle L(t), \varphi\rangle$ as $\varepsilon \to 0$ is uniform in $t \in [0, 1]$ and the real process $\langle L, \varphi\rangle$ is mean-square continuous.

Assume that in addition

(iv) *there exists a non-decreasing continuous function F on $[0, 1]$ and a number $\gamma > 0$ such that for all $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$,*

$$\begin{aligned} &\int_{[0,1]^4} (1_{[0,t_2]^2}(s,r) - 1_{[0,t_1]^2}(s,r))(1_{[0,t_2]^2}(u,v) - 1_{[0,t_1]^2}(u,v))G_\varphi(s,r,u,v) \, \mathrm{d}s \, \mathrm{d}r \, \mathrm{d}u \, \mathrm{d}v \\ &\leqslant C(\varphi)(F(t_2) - F(t_1))^{1+\gamma}, \end{aligned} \tag{2.8}$$

where $C(\varphi)$ is a positive constant depending only on φ.

Then the SILT L is a continuous $\mathscr{S}'(\mathbb{R}^d)$-process and moreover it is the weak limit in $C([0,1],\mathscr{S}'(\mathbb{R}^d))$ as $\varepsilon \to 0$ of the process L^f_ε defined by (2.3).

Note that condition (2.8) implies condition (2.7).

Concerning non-existence of SILT, the following result is obvious.

Corollary 2.5. *In the notation of Theorem* 2.4, *if $J_{s,r,u,v}$ satisfies* (i) *but*

$$\lim_{\varepsilon\to 0}\int_{[0,t]^4} J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi},\Phi^f_{\varepsilon,\varphi})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v=\infty$$

for some $\varphi\in\mathscr{S}(\mathbb{R}^d)$, $f\in\mathscr{F}$, and $t\in(0,1]$, then X does not have SILT.

Theorem 2.4 and Corollary 2.5 being quite general, their application to concrete examples requires some additional work, as it will be seen in the following sections.

We close this section with a technical lemma which will be useful to determine existence and continuity of SILT.

Lemma 2.6. *Conditions* (2.7) *and* (2.8) *are satisfied by any of the following forms of G_φ, where $C(\varphi)$ is some positive constant depending only on φ:*

(a) $$G_\varphi(s,r,u,v)=C(\varphi)((|s-u|+|r-v|)^{-\beta}+(|s-v|+|r-u|)^{-\beta}),\quad 0<\beta<2.$$

(b) $$G_\varphi(s,r,u,v)=C(\varphi)(s+r+u+v)^{-\beta},\ 0<\beta<4.$$

(c) $$G_\varphi(s,r,u,v)=C(\varphi)\left(\int_0^{s\wedge u}\int_0^{r\wedge v}(s+r+u+v-2\tau-2\sigma)^{-\beta}\mathrm{d}\sigma\,\mathrm{d}\tau\right.$$
$$\left.+\int_0^{s\wedge v}\int_0^{r\wedge u}(s+r+u+v-2\tau-2\sigma)^{-\beta}\mathrm{d}\sigma\,\mathrm{d}\tau\right),\quad 0<\beta<4.$$

Moreover, if $\beta\geqslant 2$ in the case (a) *or $\beta\geqslant 4$ in the cases* (b) *and* (c), *then condition* (2.7) *is not satisfied; more precisely, for each $t>0$,*

$$\int_{[0,t]^4} G_\varphi(s,r,u,v)\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v=\infty. \tag{2.9}$$

3. Examples

3.1. *Wiener $\mathscr{S}'(\mathbb{R}^d)$-processes*

We begin with the "simplest" Gaussian $\mathscr{S}'$-processes, i.e., the Wiener processes. Recall that a continuous centered Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process $W=(W_t)_{t\in[0,1]}$ is called a (inhomogeneous) Wiener $\mathscr{S}'(\mathbb{R}^d)$-process if its covariance functional $K(s,\varphi;t,\psi)=E(\langle W_s,\varphi\rangle\langle W_t,\psi\rangle)$ has the form

$$K(s,\varphi;t,\psi)=\int_0^{s\wedge t} q_r(\varphi,\psi)\,\mathrm{d}r,$$

where q_r is a continuous Hilbertian seminorm on $\mathscr{S}(\mathbb{R}^d)$ for each $r \in [0,1]$ (and $q_r(\cdot,\cdot)$ denotes the corresponding inner product), such that the function $r \mapsto q_r(\varphi,\psi)$ is Borel-measurable and bounded for each $\varphi,\psi \in \mathscr{S}(\mathbb{R}^d)$. The Wiener process W is said to be associated to the family of seminorms $(q_r)_{r\in[0,1]}$ (see Bojdecki and Gorostiza, 1986, Bojdecki and Jakubowski, 1989, 1990, for more information).

Given a continuous Hilbertian seminorm q on $\mathscr{S}(\mathbb{R}^d)$, we denote by $\mathscr{S}_q(\mathbb{R}^d)$ the completion of $\mathscr{S}(\mathbb{R}^d)$ with respect to q, and if $\overline{q}$ is another continuous Hilbertian seminorm on $\mathscr{S}(\mathbb{R}^d)$, then the inner product in the space $\mathscr{S}_q(\mathbb{R}^d)\widehat{\otimes}_2\mathscr{S}_{\overline{q}}(\mathbb{R}^d)$, the Hilbert (or Hilbert–Schmidt) closure of the tensor product $\mathscr{S}_q(\mathbb{R}^d)\otimes\mathscr{S}_{\overline{q}}(\mathbb{R}^d)$, will be denoted by $(\cdot,\cdot)_{q\otimes\overline{q}}$. This inner product can be computed as follows. Fix orthonormal bases $(e_k(q))_k$, $(e_l(\overline{q}))_l$ in $\mathscr{S}_q(\mathbb{R}^d), \mathscr{S}_{\overline{q}}(\mathbb{R}^d)$, respectively. It is clear that for any $\Phi \in \mathscr{S}(\mathbb{R}^{2d})$ the function $x \mapsto q(\Phi(x,\cdot),e')$, denoted by $q^y(\Phi,e')$, belongs to $\mathscr{S}(\mathbb{R}^d)$ for each $e' \in \mathscr{S}_q(\mathbb{R}^d)$, the function $y \mapsto \overline{q}(\Phi(\cdot,y),e'')$, denoted by $\overline{q}^x(\Phi,e'')$, belongs to $\mathscr{S}(\mathbb{R}^d)$ for any $e'' \in \mathscr{S}_{\overline{q}}(\mathbb{R}^d)$, and the functional $\Phi \mapsto \overline{q}^x(q^y(\Phi,e'),e'')$ belongs to $\mathscr{S}'(\mathbb{R}^{2d})$ for every $e' \in \mathscr{S}_q(\mathbb{R}^d), e'' \in \mathscr{S}_{\overline{q}}(\mathbb{R}^d)$. This and the well-known fact that $(e_k(q)\otimes e_l(\overline{q}))_{k,l}$ constitutes an orthonormal basis in $\mathscr{S}_q(\mathbb{R}^d)\widehat{\otimes}_2\mathscr{S}_{\overline{q}}(\mathbb{R}^d)$ imply the following result.

Lemma 3.1.1. *Let $q,\overline{q},(e_k(q))_k,(e_l(\overline{q}))_l$ be as above. Then, for each $\Phi,\overline{\Phi} \in \mathscr{S}(\mathbb{R}^{2d})$,*

$$\begin{aligned}(\Phi,\overline{\Phi})_{q\otimes\overline{q}} &= \sum_k \overline{q}^x(q^y(\Phi,e_k(q)),q^y(\overline{\Phi},e_k(q)))\\ &= \sum_l q^y(\overline{q}^x(\Phi,e_l(\overline{q})),\overline{q}^x(\overline{\Phi},e_l(\overline{q}))).\end{aligned} \tag{3.1.1}$$

One more notation. We denote by S the operator acting on functions $\Phi : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ in the following way:

$$S\Phi(x,y) = \Phi(y,x), \quad x,y \in \mathbb{R}^d. \tag{3.1.2}$$

Proposition 3.1.2. *Let $W = (W_t)_{t\in[0,1]}$ be a Wiener $\mathscr{S}'(\mathbb{R}^d)$-process associated with a family of seminorms $(q_r)_{r\in[0,1]}$. Then the functional J appearing in Theorem 2.4 has the form*

$$J_{s,r,u,v}(\Phi,\overline{\Phi}) = \int_0^{s\wedge u}\int_0^{r\wedge v}(\Phi,\overline{\Phi})_{q_\tau\otimes q_\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^{s\wedge v}\int_0^{r\wedge u}(\Phi,S\overline{\Phi})_{q_\tau\otimes q_\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau, \tag{3.1.3}$$

$\Phi,\overline{\Phi} \in \mathscr{S}(\mathbb{R}^{2d})$, $s,r,u,v \in [0,1]$, and it satisfies condition (i) *of that theorem.*

W has SILT if and only if the limit

$$\begin{aligned}\lim_{\varepsilon,\delta\to 0}\Bigg(&\int_{[0,t]^4}\int_0^{s\wedge u}\int_0^{r\wedge v}(\Phi^f_{\varepsilon,\varphi},\Phi^{\overline{f}}_{\delta,\varphi})_{q_\tau\otimes q_\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\\ &+\int_{[0,t]^4}\int_0^{s\wedge v}\int_0^{r\wedge u}(\Phi^f_{\varepsilon,\varphi},S\Phi^{\overline{f}}_{\delta,\varphi})_{q_\tau\otimes q_\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\Bigg)\end{aligned}$$

exists and is finite for each $t \in (0,1]$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $f,\overline{f} \in \mathscr{F}$, and it does not depend on $f,\overline{f}$.

Example 3.1.3.

(1) Let

$$q_r \equiv q, \; q(\varphi, \psi) = \int_{\mathbb{R}^d} \varphi(x)\psi(x)\mathrm{d}x.$$

In this case the Wiener process is called standard Wiener $\mathscr{S}'(\mathbb{R}^d)$-process. Formula (3.1.3) takes the form

$$\begin{aligned} J_{s,r,u,v}(\Phi, \overline{\Phi}) &= (s \wedge u)(r \wedge v) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi(x,y)\overline{\Phi}(x,y)\mathrm{d}x\,\mathrm{d}y \\ &\quad + (s \wedge v)(r \wedge u) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi(x,y)\overline{\Phi}(y,x)\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Putting $\Phi = \overline{\Phi} = \Phi^f_{\varepsilon,\varphi}$, defined by (2.2) for any $f \in \mathscr{F}$, we see that for $\varphi \neq 0$ a finite limit as $\varepsilon \to 0$ never exists, and therefore by Corollary 2.5 the standard Wiener $\mathscr{S}'(\mathbb{R}^d)$-process does not have SILT for any dimension d.

(2) Fix n different points $x_1, \ldots, x_n \in \mathbb{R}^d$ and let

$$q_r \equiv q, \; q(\varphi, \psi) = \sum_{i=1}^{n} \varphi(x_i)\psi(x_i).$$

Observe that a realization of the corresponding Wiener $\mathscr{S}'(\mathbb{R}^d)$-process is given by

$$W_t = \sum_{i=1}^{n} B_i(t)\delta_{x_i},$$

where $B_1, \ldots, B_n$ are independent one-dimensional Brownian motions and δ_{x_i} denotes the Dirac distribution at x_i. We have

$$\begin{aligned} J_{s,r,u,v}(\Phi, \overline{\Phi}) &= (s \wedge u)(r \wedge v) \sum_{i,j=1}^{n} \Phi(x_i, x_j)\overline{\Phi}(x_i, x_j) \\ &\quad + (s \wedge v)(r \wedge u) \sum_{i,j=1}^{n} \Phi(x_i, x_j)\overline{\Phi}(x_j, x_i), \end{aligned}$$

and it is seen that also in this case the Wiener $\mathscr{S}'(\mathbb{R}^d)$-process does not have SILT for any dimension d and any n.

(3) It can be also easily verified that the inhomogeneous Wiener $\mathscr{S}'(\mathbb{R}^d)$-processes occurring in the examples in Bojdecki and Gorostiza (1986), related to fluctuation limits of some particle systems, do not have SILT. On the other hand, the fluctuation limits themselves are Ornstein–Uhlenbeck $\mathscr{S}'(\mathbb{R}^d)$-processes and they have SILT for some (low) dimensions. Some of these examples are included in Section 3.3.

An example of a Wiener $\mathscr{S}'(\mathbb{R}^d)$-process for which SILT does exist is given in the next result.

Proposition 3.1.4. *Let A be a Hilbert–Schmidt operator from $L^2(\mathbb{R}^d)$ into itself, and let*

$$q_r \equiv q, \quad q(\varphi, \psi) = \int_{\mathbb{R}^d} (A\varphi)(x)(A\psi)(x)\;\mathrm{d}x.$$

Then for any dimension d the corresponding Wiener $\mathscr{S}'(\mathbb{R}^d)$-process has SILT, which is a continuous $\mathscr{S}'(\mathbb{R}^d)$-process.

Remark 3.1.5. (a) It is clear that the Wiener process in Proposition 3.1.4 lives in $L^2(\mathbb{R}^d)$. In fact, this is the general form of a homogeneous Wiener process in the Hilbert space $L^2(\mathbb{R}^d)$. We have not been able to find any other type of example of a homogeneous Wiener $\mathscr{S}'(\mathbb{R}^d)$-process which has SILT.

(b) It is worthwhile to remark that Definition 2.3 is not a generalization of the classical self-intersection local time for a finite-dimensional Wiener process (see, e.g., the references in the recent paper by Imkeller et al. 1995). Indeed, there exist at least two natural ways to embed a standard n-dimensional Wiener process $B = (B_1, \dots, B_n)$ into $\mathscr{S}'(\mathbb{R}^d)$. The first one is given by Example 3.1.3(2), and the second one consists in regarding B as a particular case of the process considered in Proposition 3.1.4: Fix any orthonormal system $e_1, \dots, e_n$ in $L^2(\mathbb{R}^d)$ and consider the Wiener process $\sum_{i=1}^n B_i e_i$. Note that A is the projection onto $span\{e_1, \dots, e_n\}$. We have seen that (independently of dimension) in the latter case SILT exists and in the former one it does not.

3.2. Three common types of Ornstein–Uhlenbeck $\mathscr{S}'(\mathbb{R}^d)$-processes

Fix any $\alpha \in (0,2]$ and denote by $p_t(x,y) \equiv p_t(x-y)$ $(t \in (0,1],\ x, y \in \mathbb{R}^d)$ the transition density of the standard symmetric α-stable process in $\mathbb{R}^d$, and by $(T_t)_{t\in[0,1]}$ the corresponding semigroup. We recall the self-similarity property of $p_t(x)$, i.e., $p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x)$, which will be used several times.

We will formulate SILT existence and continuity results for some general types of Ornstein–Uhlenbeck $\mathscr{S}'(\mathbb{R}^d)$-processes associated with (T_t) which occur in applications. They are Ornstein–Uhlenbeck processes in the sense that they are Markovian and obey equations of Langevin type (see Bojdecki and Gorostiza, 1986). Concrete examples will be given in Section 3.3.

In what follows, m denotes a measurable bounded non-negative function on $[0,1]\times\mathbb{R}^d$ for which there exist $t_m \in (0,1]$, $c_m > 0$, $\Gamma_m \in \mathscr{B}(\mathbb{R}^d)$ such that $\lambda(\Gamma_m) > 0$ (λ being the Lebesgue measure on $\mathbb{R}^d$) and

$$m(t,x) > c_m \quad \text{for} \quad t \in (0,t_m],\ x \in \Gamma_m. \tag{3.2.1}$$

Proposition 3.2.1. *Let X be a continuous centered Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process.*

(a) *If the covariance functional of X has the form*

$$K(s,\varphi;t,\psi) = \int_{\mathbb{R}^d} m(s\wedge t,x)\varphi(x)T_{|t-s|}\psi(x)\,\mathrm{d}x, \tag{3.2.2}$$

then X has SILT if and only if $d < 2\alpha$.

(b) *If the covariance functional of X has one of the forms*

$$K(s,\varphi;t,\psi) = \int_{\mathbb{R}^d} m(s\wedge t,x)T_s\varphi(x)T_t\psi(x)\,\mathrm{d}x \tag{3.2.3}$$

or

$$K(s,\varphi;t,\psi) = \int_0^{s\wedge t} \int_{\mathbb{R}^d} m(r,x) T_{s-r}\varphi(x) T_{t-r}\psi(x)\,\mathrm{d}x\,\mathrm{d}r, \tag{3.2.4}$$

then X has SILT if and only if $d < 4\alpha$.

In any of these cases (i.e. (3.2.2), (3.2.3) *or* (3.2.4)) *if SILT exists, it is a continuous $\mathscr{S}'(\mathbb{R}^d)$-process.*

Remark 3.2.2. Some of the applications below require slightly more general functions: $m(s,t,x)$ in (3.2.2) and (3.2.3), and $m(s,t,r,x)$ in (3.2.4). However, the main idea is contained in the assumption on $m(t,x)$ and we do not wish to complicate the notation further. See also Remarks 6.2.2(b) and 6.2.4.

Part (a) of Proposition 3.2.1 generalizes a result of Adler et al. (1991). They have $m \equiv 1$ and prove the cadlag property of SILT in the Brownian case ($\alpha = 2$) by means of a particle picture.

The following result will be used in the applications in the next subsection.

Proposition 3.2.3. *Let X be a continuous centered Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process with covariance functional*

$$K = K_1 + K_2,$$

where each of K_1 and K_2 has one of the forms (3.2.2), (3.2.3) *or* (3.2.4), *with the same α. Then X has SILT if and only if both processes determined by the covariance functionals K_1 and K_2 have SILT. If X has SILT, it is a continuous process.*

3.3. Specific examples of Ornstein–Uhlenbeck $\mathscr{S}'(\mathbb{R}^d)$-processes

The following concrete examples are covered by the formulation in Section 3.2, and the results are obtained by the application of Propositions 3.2.1 and 3.2.3.

3.3.1. Poisson system of α-stable processes

This model is essentially the one that leads to the α-stable (including Brownian) density process for which $\mathbb{A}$ have studied SILT. It is a Poisson system of independent symmetric α-stable processes in $\mathbb{R}^d$ with immigration ($\mathbb{A}$ do not consider immigration). The high-density fluctuation limit is a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X with covariance functional

$$K(s,\varphi;t,\psi) = (\gamma + \beta s) \int_{\mathbb{R}^d} \varphi(x) T_{t-s}\psi(x)\,\mathrm{d}x, \quad s \leqslant t.$$

The constants $\gamma > 0$ and $\beta \geqslant 0$ refer to the intensities of the initial Poisson particle field and the immigration Poisson particle field, respectively ($\beta = 0$ if there is no immigration). X is called α-stable density process in the case $\beta = 0$. This case was first studied by Martin-Löf (1976) (see also Walsh, 1986). $\mathbb{A}$ give a slightly different formulation making use of Rademacher random variables. In the case $\beta = 0$, $\mathbb{A}$ proved existence of SILT for $d < 2\alpha$ and cadlag property of the paths for $\alpha = 2$ (Adler et al. 1991, Theorems 2.2, 3.2; Adler and Rosen, 1993, Theorem 1.3). The case with immigration is contained in Bojdecki and Gorostiza (1986).

By Proposition 3.2.1(a), the process X has SILT for $d < 2\alpha$ and SILT is a continuous process, and X does not have SILT for $d \geqslant 2\alpha$.

In this model it is also possible to consider the case of immigration only (i.e., $\gamma = 0$ and $\beta > 0$). This does not satisfy assumption (3.2.1) on the function m, but this assumption can be generalized to cover also this case, and the results are the same. We preferred to avoid the additional technicalities involved in this situation.

The Brownian density process appears also in a very different context. It arises as the fluctuation limit of the density field of a stochastic lattice gas with hard core exclusion in thermal equilibrium (cf. Spohn, 1991). Note that in this case the approximating system does not quite correspond to a "particle picture".

3.3.2. System of independent α-stable processes

This is basically the model studied by Itô (1983) (see also Bojdecki and Gorostiza, 1986). It is a sequence of independent symmetric α-stable processes in $\mathbb{R}^d$ ($\alpha = 2$ in Itô, 1983) with the same initial distribution μ. The functional central limit theorem for the system yields a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X with covariance functional

$$K(s,\varphi;t,\psi) = \int_{\mathbb{R}^d} \varphi(x) T_{t-s}\psi(x) T_s^* \mu(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} \varphi(x) T_s^* \mu(x)\mathrm{d}x \int_{\mathbb{R}^d} T_{t-s}\psi(x) T_s^* \mu(x)\,\mathrm{d}x, \quad s \leqslant t,$$

where T_s^* denotes the adjoint of T_s (acting on measures). The first term is a covariance of type (3.2.2), although the function $T_s^*\mu(x)$ does not necessarily satisfy assumption (3.2.1). The second term is similar to a covariance of type (3.2.2). The same scheme of Propositions 3.2.1(a) and 3.2.3 can be used to show that the process X has SILT for $d < 2\alpha$ and SILT is a continuous process, and X does not have SILT for $d \geqslant 2\alpha$.

The previous two examples have the same SILT result, but the models are different. There is a slight confusion between the two models in Adler (1993), where it is stated that both fluctuation limit processes satisfy the same stochastic partial differential equation (with different initial conditions), but this is not so. The two equations appear in Bojdecki and Gorostiza (1986) (Examples 1 and 2). The diffusion terms coincide but the Wiener $\mathscr{S}'$-processes corresponding to the driving terms are different.

3.3.3. System of branching α-stable processes

This model consists simply in allowing the particles in the model in Example 3.3.1 to branch independently at exponentially distributed times (Bojdecki and Gorostiza, 1986, Gorostiza, 1983). Assuming finite variance branching, the high-density fluctuation limit of this system is a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X with covariance functional

$$K(s,\varphi;t,\psi)=\gamma\left[\int_{\mathbb{R}^d}e^{as}\varphi(x)T^a_{t-s}\psi(x)\,dx+m_2V\int_0^s\int_{\mathbb{R}^d}e^{ar}\varphi(x)T^a_{t+s-2r}\psi(x)\,dx\,dr\right]$$
$$+\beta\left[\frac{e^{as}-1}{a}\int_{\mathbb{R}^d}\varphi(x)T^a_{t-s}\psi(x)\,dx\right.$$
$$\left.+m_2V\int_0^s\int_{\mathbb{R}^d}\frac{e^{ar}-1}{a}\varphi(x)T^a_{t+s-2r}\psi(x)\,dx\,dr\right],\quad s\leqslant t,$$

where γ and β are again the initial and immigration Poisson intensities, respectively, V is the branching rate, a is the Malthusian parameter, m_2 is the second factorial moment of the branching law, and $T^a_t=e^{at}T_t$.

This covariance is a sum of a term of type (3.2.2) an one of type (3.2.4). (Remark 3.2.2 is relevant in this case). Proposition 3.2.3 implies that the term of type (3.2.2) dominates the self-intersection behavior, and therefore by Proposition 3.2.1(a) the process X has SILT for $d<2\alpha$ and SILT is a continuous process, and X does not have SILT for $d\geqslant 2\alpha$.

As in Example 3.3.1, the case of immigration only ($\gamma=0,\beta>0$) can also be considered and it gives the same results.

3.3.4. System of critical branching α-stable processes

This model is the same as in Example 3.3.3 in the critical case ($a=0$). Assuming $d>\alpha$, under an appropriate space-time rescaling and normalization the fluctuation limit is a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X with covariance functional

$$K(s,\varphi;t,\psi)=\gamma m_2V\int_0^s\int_{\mathbb{R}^d}\varphi(x)T_{t+s-2r}\psi(x)\,dx\,dr$$
$$+\beta m_2V\int_0^s\int_{\mathbb{R}^d}r\varphi(x)T_{t+s-2r}\psi(x)\,dx\,dr,\quad s\leqslant t,$$

where γ,β,m_2 and V are the same as above. This limit was obtained by Holley and Stroock (1981) and by Dawson (1977) in the case $\beta=0$ (see Bojdecki and Gorostiza, 1986, Example 4, for the case $\beta>0$).

The covariance is of type (3.2.4). Hence, by Proposition 3.2.1(b) the process X has SILT for $d<4\alpha$ and SILT is a continuous process, and X does not have SILT for $d\geqslant 4\alpha$.

Note that in this model the space-time rescaling has the effect of increasing towards infinity both the intensity of the particles and the intensity of the branching, in contrast to Example 3.3.3, where the intensity of the particles grows but the intensity of the branching remains bounded.

3.3.5. System of supercritical branching α-stable processes

This model is a supercritical branching particle system that has been studied in several papers, the last one being Fernández and Gorostiza (1991) (in the case $\alpha=2$). The model and the process are difficult to explain in a few lines, but it is not necessary for our present purpose, so we avoid it. The space-time rescaling fluctuation limit is a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X with covariance functional

$$K(s,\varphi;t,\psi)=c\left(\gamma\int_B T_s\varphi(x)T_t\psi(x)\,\mathrm{d}x\right.$$
$$\left.+\beta\int_0^s\int_Q T_{s-r}\varphi(x)T_{t-r}\psi(x)\mathrm{d}x\,\mathrm{d}r\right),\quad s\leqslant t,$$

where c is a positive constant, γ and β are initial and immigration Poisson intensities (as in some of the previous examples), and B and Q are Borel subsets of $\mathbb{R}^d$ (of positive Lebesgue measure).

The covariance is a sum of a term of type (3.2.3) and another of type (3.2.4). By Propositions 3.2.1(b) and 3.2.3, X has SILT for $d<4\alpha$ and SILT is a continuous process, and X does not have SILT for $d\geqslant 4\alpha$.

Note that in this case the intensity of the branching also goes to infinity, as in the previous example.

3.3.6. *Voter model*

The fluctuation limit, under a space-time rescaling and normalization, of the magnetization field of the voter model with symmetric nearest-neighbor interaction for dimension $d\geqslant 3$ was obtained by Presutti and Spohn (1983). It is a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X with covariance functional

$$K(s,\varphi;t,\psi)=c\int_0^s\int_{\mathbb{R}^d}[1-(T_rm(x))^2]T_{s-r}\varphi(x)T_{t-r}\psi(x)\,\mathrm{d}x\,\mathrm{d}r,\quad s\leqslant t,$$

where c is a positive constant, m is a continuous function from $\mathbb{R}^d$ into $[-1,1]$ (the case $|m|=1$ is excluded), and (T_t) is the Brownian semigroup.

This covariance is of type (3.2.4), and therefore by Proposition 3.2.1(b) the process X has SILT for $d\leqslant 7$ and the SILT process is continuous, and X does not have SILT for $d\geqslant 8$.

3.4. *Other examples of Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes*

We give here examples of processes with covariances that are not of the types in Section 3.2.

3.4.1. *Poisson system of α-stable bridges*

This model, which has been studied by Gorostiza (1994), is like the one in Example 3.3.1 (without immigration), except that each particle undergoes a bridge leading to its starting point. The high-density fluctuation limit is a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process X with covariance functional

$$K(s,\varphi;t,\psi)=\int_{\mathbb{R}^d}\varphi(x)T_{(t-s)[1-(t-s)]}\psi(x)\,\mathrm{d}x,\quad s\leqslant t.$$

Note that this process is not of Ornstein–Uhlenbeck type (it is not Markovian).

X has SILT for $d < 2\alpha$ and the SILT process is continuous. This result is not surprising because for small s and t the covariance is almost like the one in Example 3.3.1.

3.4.2. Two-level branching system

This example will be treated in detail because it requires a more delicate analysis than the previous ones and it gives a new result (existence of SILT up to dimension 11 in one case). It concerns a two-level branching system studied by Gorostiza (1994) (see also Gorostiza, 1995, for the Brownian case). The model is a system of branching superprocesses, the effect of the first level branching being contained in the superprocesses (see Dawson, 1993, for basic background on superprocesses). The high-density fluctuation limit of the aggregation of the system is a Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process. In fact, it is an Ornstein–Uhlenbeck $\mathscr{S}'(\mathbb{R}^d)$-process, but the covariance is not of the types considered in Section 3.2. In order to discuss the SILT results, in particular in connection with dimension gaps (Section 4), we will give a rough description of the model (see Gorostiza, 1994, for a detailed description of a more general model and motivations).

Let $\mathscr{M}_p(\mathbb{R}^d)$ denote the space of p-tempered measures on $\mathbb{R}^d$, i.e., Radon measures μ such that $\langle \mu, \varphi_p\rangle < \infty$, where $\varphi_p(x) = (1+\|x\|^2)^{-p}$, with $p > d/2$ and in addition $p < (d+\alpha)/2$ if $\alpha < 2$, and $\alpha \in (0,2]$ is the index of the underlying symmetric stable particle motion (here $\langle \mu, \varphi\rangle = \int \varphi \, d\mu$). $\mathscr{M}_p(\mathbb{R}^d)$ carries the p-vague topology (defined by φ_p and the non-negative continuous functions with compact support). For each $n = 1, 2, \ldots$, let N^n denote a Poisson random measure on $\mathscr{M}_p(\mathbb{R}^d)$ with mean measure nR, where R is a locally finite measure on $\mathscr{M}_p(\mathbb{R}^d)$. The points of N^n are regarded as measure-valued particles, or "superparticles" for short. As time elapses, each superparticle independently migrates in $\mathscr{M}_p(\mathbb{R}^d)$ according to a critical (finite variance) super α-stable process, and at an exponentially distributed time it produces offspring at its own site according to a critical (finite variance) branching law. The new superparticles behave in the same manner. Note that the superparticle motion does not die out under the assumption $d > \alpha$ to be made below.

Let $B_{ij}(t)$ denote the location in $\mathscr{M}_p(\mathbb{R}^d)$ of the jth superparticle in the ith branching superprocess at time t, and let $Y_t^n = \sum_{ij} B_{ij}(t)$, where the index n refers to the initial Poisson measure (with mean nR). The $\mathscr{M}_p(\mathbb{R}^d)$-valued Markov process $Y^n = (Y_t^n)_{t\geqslant 0}$, called the aggregation, is the subject of Gorostiza (1994). In particular, the fluctuation limit as $n \to \infty$ of Y^n (normalized by $n^{1/2}$) is identified as an Ornstein–Uhlenbeck $\mathscr{S}'$-process. We consider here the special case where the measure R on $\mathscr{M}_p(\mathbb{R}^d)$ which defines the initial Poisson mean is R_∞, the canonical measure of the (infinitely divisible) equilibrium state of a critical (finite variance) super α-stable process (the intensity of R_∞ on $\mathbb{R}^d$ is the Lebesgue measure). Such non-trivial equilibrium states are known to exist for dimensions $d > \alpha$ (see Dawson, 1993; Dawson and Perkins, 1991; Gorostiza and Wakolbinger, 1991), which we assume henceforth. In this case the fluctuation limit of Y^n as $n \to \infty$ is a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process $X = (X_t)_{t\geqslant 0}$ with covariance functional

$$K(s,\varphi;t,\psi) = \sum_{i=1}^{3} K_i(s,\varphi;t,\psi),$$

where, for $s \leqslant t$,

$$K_1(s,\varphi;t,\psi) = c\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{T_s\varphi(x)T_t\psi(y)}{\|x-y\|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y,$$

$$K_2(s,\varphi;t,\psi) = V_1\int_0^s\int_{\mathbb{R}^d} \varphi(x)T_{s+t-2r}\psi(x)\,\mathrm{d}x\,\mathrm{d}r,$$

$$K_3(s,\varphi;t,\psi) = V_1V_2\int_0^s\int_0^r\int_{\mathbb{R}^d} \varphi(x)T_{s+t-2r+2v}\psi(x)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}r,$$

(T_t) is the semigroup of the symmetric α-stable process in $\mathbb{R}^d$, and $c>0, V_1\geqslant 0, V_2\geqslant 0$ are constants. V_1 and V_2 stand for the branching rates at levels 1 and 2, respectively. The special cases $V_1=0$ and/or $V_2=0$ mean that there is no branching at the corresponding level. For $V_1=0$ the model represents a Poisson (nR_∞) system of deterministic T_t^*-flows in $\mathscr{M}_p(\mathbb{R}^d)$, because in this case the superprocess limit is a deterministic T_t^*-flow. (In this case it does not matter for the fluctuation limit if $V_2=0$ or $V_2>0$.) For $V_1>0$ and $V_2=0$ the model represents a Poisson (nR_∞) system of superprocesses. (Note: In Gorostiza (1994) R_∞ is taken as the canonical measure of the underlying superprocess, and therefore the constant c in K_1 contains the factor $V_1>0$, see formula (4.27) in Gorostiza (1994); however, R_∞ can be chosen independently of the underlying superprocess, and hence it is possible to have $V_1>0$ for R_∞ and $V_1=0$ for the superparticle motion.)

We will consider first the case of a Poisson (nR_∞) system of deterministic T_t^*-flows. The covariance functional of the Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process is given by K_1, and we denote by X_1 the process with this covariance. It turns out that this case, which is the simplest special case of the model, yields the most interesting result.

Proposition 3.4.1. *The process X_1 has SILT if and only if $2\alpha<d<6\alpha$ ($5\leqslant d\leqslant 11$ in the Brownian case), and when it exists the SILT process is continuous.*

We note that $d>2\alpha$ is also the condition for non-trivial long-time behaviors of the fluctuation limit process X (cf. Gorostiza, 1994) and of the 2–level superprocess (cf. Gorostiza et al., 1995; Wu, 1994). We also remark that the critical dimension 6α for existence of SILT for X_1 is due to a combined effect of the measure R_∞ and the T_t^*-flow, both of which are characterized by the same value of α. If we take another measure R instead of R_∞ for the Poisson mean, then the result can change. For example, taking $R=\delta_{\delta_x}\lambda(\mathrm{d}x)$, λ Lebesgue measure (cf. Wu, 1994), and the α-stable T_t^*-flow leads to critical dimension 4α instead of 6α. On the other hand, taking different values of α for R_∞ and for the T_t^*-flow also leads to a different critical dimension. (We have not carried out this computation, but it can be seen that the result must change. We leave this question for future work on systems of particles of different types.)

Now we consider the case of a Poisson (nR_∞) system of superprocesses. The covariance functional of the Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process is given by K_1+K_2, and we denote by X_2 the process with this covariance. Since K_2 is a covariance of the type (3.2.4), by Proposition 3.2.1(b) the SILT for the process corresponding to K_2 exists for dimensions $\alpha<d<4\alpha$. The result for X_2 can then be obtained by a natural generalization of Proposition 3.2.3. Finally, we consider the full process X (with covariance

$K = \sum_{i=1}^{3} K_i$). The covariance K_3 is similar to the type (3.2.4), and we can extend the procedure also to this case. Thus we have the following result.

Proposition 3.4.2. *The Gaussian $\mathscr{S}'(\mathbb{R}^d)$-processes X_2 and X have SILT if and only if $2\alpha < d < 4\alpha$ ($5 \leqslant d \leqslant 7$ in the Brownian case), and the SILT processes are continuous.*

It can be shown that for $R = \delta_{\delta_x}\lambda(\mathrm{d}x)$ the three processes X_1, X_2 and X have SILT for $d < 4\alpha$.

3.4.3. Occupation time processes

Given an $\mathscr{S}'(\mathbb{R}^d)$-process X, its occupation time process Y is defined by $Y_t = \int_0^t X_u \mathrm{d}u, t \geqslant 0$ (we borrow the definition for measure-valued processes in Iscoe, 1986). If X is centered Gaussian, then so is Y and it is continuous. Note that in general Y is not an Ornstein–Uhlenbeck process (it is not Markovian with respect to its own filtration).

Proposition 3.4.3. *If X satisfies conditions* (i) *with J replaced by its absolute value,* (ii) *and* (iii) *of Theorem 2.4, then the SILT of Y exists and it is a continuous $\mathscr{S}'(\mathbb{R}^d)$-process.*

Note that condition (i) with $|J|$ holds in all our examples.

4. Some comments and questions on dimension gaps

We consider the Brownian case in order to compare with $\mathbb{A}$ (similar comments can be made for the α-stable case). As explained by Adler (1993) , in $\mathbb{R}^d$ independent Brownian motions have intersection local time (ILT) for $d \leqslant 3$ and the Brownian density process has SILT also for $d \leqslant 3$, and the latter result occurs because the SILT for the density process is obtained by means of a particle picture as a limit of a sum of ILTs of pairs of independent Brownian motions. In Adler, et al. (1991) this is considered as the justification for calling (1.1) a SILT for the density process. Hence, the increased intensity of particles does not generate a dimension gap. On the other hand, also as explained by Adler (1993), the super Brownian process has SILT for $d \leqslant 5$, and this gap is a consequence of the clumping phenomenon associated with the branching inherent in the superprocess. (Recent papers on SILT for superprocesses are Adler and Lewin (1992) and Rosen (1992)).

Example 3.3.3 shows that if branching is allowed in the system of Brownian motions, but without increasing the branching rate, then the high-density fluctuation limit process (which corresponds to the density process) also has SILT for $d \leqslant 3$. Thus, adding branching does not generate a dimension gap if the branching intensity remains bounded. In contrast, Example 3.3.4 shows that under a rescaling which increases towards infinity the (critical) branching intensity, the fluctuation limit process has SILT for $d \leqslant 7$. Note that the critical dimension 7 also occurs in the supercritical branching model in Example 3.3.5, where the intensity of the branching also goes to infinity. Hence, the clumping phenomenon of the branching generates a dimension gap only under high intensity of branching. Moreover, this gap is bigger than the one associated with the superprocess limit. How can this difference be explained?

The previous comments refer to systems of particles in $\mathbb{R}^d$. New things happen for a system of (super) particles in $\mathscr{M}_p(\mathbb{R}^d)$. The graphs of two independent super Brownian motions can intersect for $d \leqslant 5$ (cf. Barlow, et al., 1991). On the other hand, as noted at the end of Example 3.4.2, the high-density fluctuation limit of a Poisson $(n\delta_{\delta_x}\lambda(\mathrm{d}x))$ system of super Brownian motions (i.e., a "super Brownian density process") has SILT for $d \leqslant 7$, not for $d \leqslant 5$ as one would expect from Barlow, et al. (1991) by analogy with the argument of Adler (1993), because the graphs of the pairs of super Brownian motions must intersect in order for the fluctuation limit to have SILT, and there is no branching in the model. Of course, there is the branching inherent in the superprocesses, but the point is that the superprocess paths do not branch. So in this case the dimension gap is not caused by branching of the paths. How can it be explained?

Another interesting effect is the following. Example 3.4.2 shows that the high-density fluctuation limit of a Poisson (nR_∞) system of deterministic (Brownian) T_t^*-flows (without branching of the paths) has SILT for $5 \leqslant d \leqslant 11$, but if the T_t^*-flows are replaced by super Brownian motions (which contain a built-in continuous branching), then the high-density fluctuation limit process has SILT for $5 \leqslant d \leqslant 7$. Hence, there is a dimension gap, but now it is a decrease, in contrast to the previous gaps. What is the explanation for this gap?

These questions about dimension gaps arise naturally from some of our branching examples, but since they are not our objective in this paper and we do not have good answers yet, we leave them for the future.

5. Proofs for Section 2

Throughout the proofs in this and the following section we will denote generic positive constants by $C, C_1, C_2, \ldots$, by $C(\varphi), C_1(\varphi), C_2(\varphi), \ldots$ when they depend (only) on φ, and similarly for other such dependences.

For two real Hilbert spaces H_1 and H_2, we denote by $H_1 \widehat{\otimes}_1 H_2$ the nuclear completion of the tensor product $H_1 \otimes H_2$ (see, e.g., Neveu, 1968). The following lemma is most likely well-known to the specialists but we have not been able to find a reference for it, so we present a short proof.

Lemma 5.1. *Let q be a continuous Hilbertian seminorm on $\mathscr{S}(\mathbb{R}^d)$. Then $\mathscr{S}(\mathbb{R}^{2d})$ is continuously embedded in $\mathscr{S}_q(\mathbb{R}^d)\widehat{\otimes}_1\mathscr{S}_q(\mathbb{R}^d)$, where $\mathscr{S}_q(\mathbb{R}^d)$ is the q-completion of $\mathscr{S}(\mathbb{R}^d)$. (Notation: $\mathscr{S}(\mathbb{R}^{2d}) \hookrightarrow \mathscr{S}_q(\mathbb{R}^d)\widehat{\otimes}_1\mathscr{S}_q(\mathbb{R}^d)$.)*

Proof. Let $\|\cdot\|_{q(1)}$ denote the norm on $\mathscr{S}_q(\mathbb{R}^d)\widehat{\otimes}_1\mathscr{S}_q(\mathbb{R}^d)$. First observe that if $\overline{q}$ is another continuous Hilbertian seminorm on $\mathscr{S}(\mathbb{R}^d)$ and q is continuous with respect to $\overline{q}$, then, clearly, $\mathscr{S}_{\overline{q}}\widehat{\otimes}_1 S_{\overline{q}} \hookrightarrow \mathscr{S}_q\widehat{\otimes}_1\mathscr{S}_q$. Indeed, it suffices to recall that

$$\| \Phi \|_{q(1)} = \inf \left\{ \sum_i q(\varphi_i)q(\psi_i) : \Phi(x,y) = \sum_i \varphi_i(x)\psi_i(y) \right\} \tag{5.1}$$

for $\Phi \in \mathscr{S}(\mathbb{R}^d) \otimes \mathscr{S}(\mathbb{R}^d)$.

Let $\|\cdot\|_0 \leqslant \|\cdot\|_1 \leqslant \cdots$ be the usual Hilbertian norms determining the (nuclear Fréchet space) topology of $\mathscr{S}(\mathbb{R}^d)$. Fix m and $n, m < n$, such that q is continuous with respect to $\|\cdot\|_m$ and the embedding $\mathscr{S}_n(\mathbb{R}^d) \hookrightarrow \mathscr{S}_m(\mathbb{R}^d)$ is Hilbert–Schmidt. By the previous remark it suffices to show that $\mathscr{S}(\mathbb{R}^{2d}) \hookrightarrow \mathscr{S}_n(\mathbb{R}^d)\widehat{\otimes}_1\mathscr{S}_n(\mathbb{R}^d)$.

It is known (cf. Neveu, 1968) that for $\Phi \in \mathscr{S}_n(\mathbb{R}^d) \otimes \mathscr{S}_n(\mathbb{R}^d)$,

$$\begin{aligned} \| \Phi \|_{n(1)} = \sup \Bigg\{ & \sum_i |(\Phi, e_i \otimes h_i)_{\mathscr{S}_n(\mathbb{R}^d)\otimes\mathscr{S}_n(\mathbb{R}^d)}| \\ & : (e_i)_i, (h_i)_i \text{ orthornormal systems in } \mathscr{S}_n(\mathbb{R}^d), e_i, h_i \in \mathscr{S}(\mathbb{R}^d) \Bigg\}. \end{aligned} \tag{5.2}$$

Fix two orthonormal systems $(e_i)_i, (h_i)_i$ in $\mathscr{S}_n(\mathbb{R}^d)$, $e_i, h_i \in \mathscr{S}(\mathbb{R}^d)$, and let $\Phi \in \mathscr{S}(\mathbb{R}^d) \otimes \mathscr{S}(\mathbb{R}^d)$. Assume $d = 1$ for simplicity. Then

$$\begin{aligned} &(\Phi, e_i \otimes h_i)_{\mathscr{S}_n(\mathbb{R})\otimes\mathscr{S}_n(\mathbb{R})} \\ &= \sum_{k,l=0}^{n} \int_{\mathbb{R}^2} (1+x^2)^n(1+y^2)^n \frac{\partial^{k+l}\Phi(x,y)}{\partial x^k \partial y^l} \frac{\mathrm{d}^k e_i(x)}{\mathrm{d}x^k} \frac{\mathrm{d}^l h_i(y)}{\mathrm{d}y^l} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Every summand corresponding to $k > m$ can be represented, integrating by parts $k-m$ times, as

$$\int_{\mathbb{R}^2} (\text{a sum of products of polynomials and derivatives of } \Phi) \frac{\mathrm{d}^m e_i(x)}{\mathrm{d}x^m} \frac{\mathrm{d}^l h_i(y)}{\mathrm{d}y^l} \,\mathrm{d}x\,\mathrm{d}y,$$

and then the same can be done for $l > m$. Thus, we see that there exist a positive constant $C(m,n)$ and a natural number $N(m,n)$ such that

$$|(\Phi, e_i \otimes h_i)_{\mathscr{S}_n(\mathbb{R})\otimes\mathscr{S}_n(\mathbb{R})}| \leqslant C(n,m) ||| \Phi |||_{N(n,m)} \sum_{k,l=0}^{m} \int_{\mathbb{R}} \left| \frac{\mathrm{d}^k e_i(x)}{\mathrm{d}x^k} \right| \mathrm{d}x \int_{\mathbb{R}} \left| \frac{\mathrm{d}^l h_i(y)}{\mathrm{d}y^l} \right| \mathrm{d}y,$$

where $(|||\cdot|||_N)_N$ denotes the second basic system of ("uniform") norms defining the topology of $\mathscr{S}(\mathbb{R})$.

We have

$$\begin{aligned} \sum_{k=0}^{m} \int_{\mathbb{R}} \left| \frac{\mathrm{d}^k}{\mathrm{d}x^k} e_i(x) \right| \mathrm{d}x &\leqslant \sum_{k=0}^{m} \left(\int_{\mathbb{R}} (1+x^2)^m \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} e_i(x) \right)^2 \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}} (1+x^2)^{-m} \mathrm{d}x \right)^{1/2} \\ &\leqslant C(m) \| e_i \|_m . \end{aligned}$$

Hence,

$$\sum_i |(\Phi, e_i \otimes h_i)_{\mathscr{S}_n(\mathbb{R})\otimes\mathscr{S}_n(\mathbb{R})}| \leqslant C_1(n,m)|||\Phi|||_{N(n,m)} \left(\sum_i \|e_i\|_m^2\right)^{1/2} \left(\sum_i \|h_i\|_m^2\right)^{1/2}$$
$$\leqslant C_2(n,m)|||\Phi|||_{N(n,m)},$$

since $(\sum_i \|e_i\|_m^2)^{1/2}$ and $(\sum_i \|h_i\|_m^2)^{1/2}$ are not greater than the Hilbert–Schmidt norm of the embedding $\mathscr{S}_n(\mathbb{R}) \hookrightarrow \mathscr{S}_m(\mathbb{R})$.

By (5.2) the lemma is proved. □

Proof of Proposition 2.1. We keep the notation of Lemma 5.1 and we denote (as usual) by $\|\cdot\|_{-n}$ the norm on $\mathscr{S}'_n(\mathbb{R}^d)$.

X being a continuous Gaussian $\mathscr{S}'(\mathbb{R}^d)$-process, it is known (cf. Mitoma, 1981) that there exists an n such that X is continuous in the space $\mathscr{S}'_n(\mathbb{R}^d)$ and

$$E\left(\sup_{t\in[0,1]} \|X_t\|_{-n}^2\right) < \infty.$$

If K is the covariance functional of X we have

$$|K(s,\varphi;t,\psi)| \leqslant (E\,\|X_s\|_{-n}\|X_t\|_{-n})\,\|\varphi\|_n\|\psi\|_n$$
$$\leqslant E\left(\sup_{t\in[0,1]} \|X_t\|_{-n}^2\right) \|\varphi\|_n\|\psi\|_n\,. \tag{5.3}$$

Let $:X_s \otimes X_t: (\Phi)$ denote the right-hand side of (2.1) extended by linearity to $\mathscr{S}(\mathbb{R}^d)\otimes\mathscr{S}(\mathbb{R}^d)$. Then, by linearity, $E(:X_s\otimes X_r:(\Phi)\,:X_u\otimes X_v:(\overline{\Phi}))$ is equal to the right-hand side of (2.5) for $\Phi, \overline{\Phi}$ of the form (2.4), and (2.5), (5.1) and (5.3) yield

$$E((:X_t\otimes X_s:(\Phi))^2) \leqslant C\left(\sum_{i=1}^N \|\varphi_i\|_n\|\psi_i\|_n\right)^2$$
$$\leqslant C\,\|\Phi\|_{n(1)}^2$$

for $\Phi = \sum_{i=1}^N \varphi_i\otimes\psi_i \in \mathscr{S}(\mathbb{R}^d)\otimes\mathscr{S}(\mathbb{R}^d)$, where C is a constant not depending on s,t. Hence,

$$\int_{[0,1]^2} E((:X_t\otimes X_s:(\Phi))^2)\,\mathrm{d}s\,\mathrm{d}t \leqslant C\,\|\Phi\|_{n(1)}^2,$$

so, by Lemma 5.1, the mapping $\Phi \mapsto :X_\cdot\otimes X_\cdot:(\Phi)$ can be extended to a continuous linear functional from $\mathscr{S}(\mathbb{R}^{2d})$ into $L^2([0,1]\times[0,1]\times\Omega)$. By nuclearity of $\mathscr{S}(\mathbb{R}^{2d})$ we can apply the regularization theorem (cf. Itô, 1984) to obtain an $\mathscr{S}'(\mathbb{R}^{2d})$-valued version of this functional. □

Corollary 5.2. *Let X be as in Proposition* 2.1. *Then, for each $t\in[0,1]$, $\Phi,\overline{\Phi}\in\mathscr{S}(\mathbb{R}^{2d})$,*

$$E\left(\int_{[0,t]^4} |\langle :X_s\otimes X_r:,\Phi\rangle\langle :X_u\otimes X_v:,\overline{\Phi}\rangle|\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\right) < \infty,$$

and the mapping

$$(\Phi,\overline{\Phi}) \mapsto E\left(\int_{[0,1]^4} \langle :X_s\otimes X_r:,\Phi\rangle\langle :X_u\otimes X_v:,\overline{\Phi}\rangle \mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v\right)$$

is continuous bilinear from $\mathscr{S}(\mathbb{R}^{2d})\times\mathscr{S}(\mathbb{R}^{2d})$ *into* $\mathbb{R}$.

Proof of Theorem 2.4. Assumption (i) and Corollary 5.2 imply that the formula

$$E\left(\int_{[0,t]^4} \langle :X_s\otimes X_r:,\Phi\rangle\langle :X_u\otimes X_v:,\overline{\Phi}\rangle \mathrm{d}s\mathrm{d}r\mathrm{d}u\mathrm{d}v\right) = \int_{[0,t]^4} J_{s,r,u,v}(\Phi,\overline{\Phi})\mathrm{d}s\mathrm{d}r\mathrm{d}u\mathrm{d}v \tag{5.4}$$

holds for each $\Phi,\overline{\Phi}\in\mathscr{S}(\mathbb{R}^{2d})$.

By (2.3) and (5.4) we have, for any $f\in\mathscr{F}$, $\varphi\in\mathscr{S}(\mathbb{R}^d)$, $0\leqslant t_1<t_2\leqslant 1$,

$$\begin{aligned} &E(\langle L_\varepsilon^f(t_2),\varphi\rangle - \langle L_\varepsilon^f(t_1),\varphi\rangle)^2 \\ &= \int_{[0,1]^4} (1_{[0,t_2]^2}(s,r) - 1_{[0,t_1]^2}(s,r))(1_{[0,t_2]^2}(u,v) - 1_{[0,t_1]^2}(u,v)) \\ &\quad \times J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi},\Phi^f_{\varepsilon,\varphi})\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v. \end{aligned} \tag{5.5}$$

By assumption (iii), this implies that the process $\langle L_\varepsilon^f,\varphi\rangle$ is mean-square continuous. It is also easy to see that, by assumptions (ii) and (iii) (and of course, again by (5.4)),

$$\lim_{\varepsilon,\delta\to 0}\sup_{t\in[0,1]} E(\langle L_\varepsilon^f(t),\varphi\rangle - \langle L_\delta^{\overline{f}}(t),\varphi\rangle)^2 = 0$$

for each $f,\overline{f}\in\mathscr{F}$, $\varphi\in\mathscr{S}(\mathbb{R}^d)$. This proves that a uniform (in t) mean-square limit of $\langle L_\varepsilon^f(t),\varphi\rangle$ as $\varepsilon\to 0$ exists, is independent of f and is mean-square continuous in t. Moreover, for any fixed t this limit is linear and mean-square continuous as a function of $\varphi\in\mathscr{S}(\mathbb{R}^d)$ ($\mathscr{S}(\mathbb{R}^d)$ being Fréchet, it is barrelled, so the Banach–Steinhaus theorem applies).

Hence, to complete the proof of the first part of the theorem it suffices to apply the regularization theorem (cf. Itô, 1984).

To prove continuity of SILT paths in $\mathscr{S}'(\mathbb{R}^d)$ we need only show that the processes L_ε^f converge weakly in $C([0,1],\mathscr{S}'(\mathbb{R}^d))$ as $\varepsilon\to 0$ (by Corollary 2.2 we know that they do have paths in this space). It is clear that the finite-dimensional distributions converge (by Definition 2.3 and the first part of the theorem), so by Mitoma's theorem (Mitoma, 1983) it suffices to prove that for any fixed $\varphi\in\mathscr{S}(\mathbb{R}^d)$ the family of processes $\{\langle L_\varepsilon^f,\varphi\rangle : 0<\varepsilon<1\}$ is tight in $C([0,1],\mathbb{R})$. But this is immediate since by (5.5), (2.6) and (2.8) we have

$$E(\langle L_\varepsilon^f(t_2),\varphi\rangle - \langle L_\varepsilon^f(t_1),\varphi\rangle)^2 \leqslant C(\varphi)(F(t_2)-F(t_1))^{1+\gamma}$$

for each $t_1, t_2 \in [0,1]$, $t_1 < t_2$, so the tightness follows from a well-known criterion (Billingsley, 1968, p. 95). □

Proof of Lemma 2.6. We have noted that condition (2.8) implies condition (2.7). However, since condition (2.7) suffices for existence of SILT it is worthwhile to verify the two conditions separately.

(a) We have $G_\varphi = G_\varphi^{(1)} + G_\varphi^{(2)}$, where

$$G_\varphi^{(1)}(s,r,u,v) = C(\varphi)(|s-u| + |r-v|)^{-\beta},$$
$$G_\varphi^{(2)}(s,r,u,v) = C(\varphi)(|s-v| + |r-u|)^{-\beta}.$$

We consider $G_\varphi^{(1)}$ only, since it is clear that $G_\varphi^{(2)}$ will yield the same result.

We have, as in Adler et al., (1991),

$$G_\varphi(s,r,u,v) \leqslant C(\varphi)|s-u|^{-\beta/2}|r-v|^{-\beta/2}, \tag{5.6}$$

and it is clear that $\int_{[0,1]^2} |s-u|^{-\beta/2} \mathrm{d}s\, \mathrm{d}u < \infty$ if $\beta < 2$, so (2.7) is satisfied. On the other hand, by straightforward calculations, $\int_{[0,t]^2} (|s-u| + |r-v|)^{-2} \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v = \infty$, and therefore (2.9) holds for any $\beta \geqslant 2$.

To prove (2.8) put $\beta/2 = 1-\gamma$, $0 < \gamma < 1$. By (5.6) the left-hand side of (2.8) is bounded above by

$$C(\varphi)\int_{[0,1]^4} (1_{[0,t_2]^2}(s,r) - 1_{[0,t_1]^2}(s,r))(1_{[0,t_2]^2}(u,v) - 1_{[0,t_1]^2}(u,v))$$
$$|s-u|^{-1+\gamma}|r-v|^{-1+\gamma} \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v.$$

The last integral can be computed directly. After quite long and tedious but completely elementary calculations we find that it is equal to

$$\begin{aligned} &C_1(\gamma)(t_2^{2+2\gamma} - 2t_2^{1+\gamma}t_1^{1+\gamma} + 2t_2^{1+\gamma}(t_2-t_1)^{1+\gamma} + t_1^{2+2\gamma} \\ &\quad + 2t_1^{1+\gamma}(t_2-t_1)^{1+\gamma} - (t_2-t_1)^{2+2\gamma}) \\ &\leqslant C_1(\gamma)((t_2^{1+\gamma} - t_1^{1+\gamma})^2 + (t_2-t_1)^{1+\gamma}(2t_2^{1+\gamma} + 2t_1^{1+\gamma})) \\ &\leqslant C_2(\gamma)(t_2-t_1)^{1+\gamma}, \end{aligned}$$

so (2.8) is proved in this case.

(b) It can be verified by direct computation that (2.7) is satisfied if $0 < \beta < 4$ and (2.9) is satisfied if $\beta \geqslant 4$. The proof of (2.8) is also very simple in this case since

$$G_\varphi(s,r,u,v) \leqslant C(\varphi)(s+r)^{-\beta/2}(u+v)^{-\beta/2},$$

and therefore, by symmetry, the left-hand side of (2.8) can be estimated by

$$\begin{aligned} &C(\varphi)\left(\int_{[0,1]^2} (1_{[0,t_2]^2}(s,r) - 1_{[0,t_1]^2}(s,r))(s+r)^{-\beta/2} \mathrm{d}s\, \mathrm{d}r\right)^2 \\ &= C(\varphi)\left(\int_{[0,t_2]^2} (s+r)^{-\beta/2} \mathrm{d}s\, \mathrm{d}r - \int_{[0,t_1]^2} (s+r)^{-\beta/2} \mathrm{d}s\, \mathrm{d}r\right)^2. \end{aligned}$$

Thus, (2.8) holds for $F(t) = \int_{[0,t]^2} (s+r)^{-\beta/2} \mathrm{d}s\, \mathrm{d}r$ (well-defined if $\beta < 4$).

(c) As in (a), we have $G_\varphi = G_\varphi^{(1)} + G_\varphi^{(2)}$, the expressions for $G_\varphi^{(1)}$ and $G_\varphi^{(2)}$ being obvious.

Note that there exist $0 < \delta < 2$ and $C_1(\varphi, \beta)$ such that

$$\begin{aligned}
G_\varphi^{(1)}(s,r,u,v) &\leqslant C_1(\varphi,\beta)\int_0^{s\wedge u}\int_0^{r\wedge v}(s+u+r+v-2\tau-2\sigma)^{-4+\delta}\mathrm{d}\sigma\,\mathrm{d}\tau\\
&= \frac{C_1(\varphi,\beta)}{4(3-\delta)(2-\delta)}((s+u+r+v)^{-2+\delta}\\
&\quad -(|s-u|+r+v)^{-2+\delta}-(s+u-|r-v|)^{-2+\delta}\\
&\qquad +(|s-u|+|r-v|)^{-2+\delta})\\
&\leqslant C_2(\varphi,\beta,\delta)(|s-u|+|r-v|)^{-2+\delta},
\end{aligned}$$

so we are led to case (a).

Finally, once more a straightforward computation shows that (2.9) holds for $\beta = 4$, and hence for $\beta > 4$ as well. □

6. Proofs for Section 3

6.1. Proofs for Section 3.1

Proof of Proposition 3.1.2. It is seen by formula (2.5) that for $\Phi, \overline{\Phi} \in \mathscr{S}(\mathbb{R}^d)\otimes\mathscr{S}(\mathbb{R}^d)$ of the form (2.4),

$$\begin{aligned}
&J_{s,r,u,v}(\Phi,\overline{\Phi})\\
&= \int_0^{s\wedge u}\int_0^{r\wedge v}\sum_{i,j} q_\tau(\varphi_i,\overline{\varphi}_j)q_\sigma(\psi_i,\overline{\psi}_j)\mathrm{d}\sigma\mathrm{d}\tau + \int_0^{s\wedge v}\int_0^{r\wedge u}\sum_{i,j} q_\tau(\varphi_i,\overline{\psi}_j)q_\sigma(\psi_i,\overline{\varphi}_j)\,\mathrm{d}\sigma\,\mathrm{d}\tau\\
&= \int_0^{s\wedge u}\int_0^{r\wedge v}(\Phi,\overline{\Phi})_{q_\tau\otimes q_\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^{s\wedge v}\int_0^{r\wedge u}(\Phi,S\overline{\Phi})_{q_\tau\otimes q_\sigma}\,\mathrm{d}\sigma\,\mathrm{d}\tau,
\end{aligned}$$

the last expression being well-defined for $\Phi, \overline{\Phi} \in \mathscr{S}(\mathbb{R}^{2d})$ and bilinear in $\Phi, \overline{\Phi}$. (Recall that S is defined by (3.1.2)). Since q_τ, q_σ are continuous seminorms in $\mathscr{S}(\mathbb{R}^d)$, it is clear that if $\Phi_n \to \Phi$, $\overline{\Phi}_n \to \overline{\Phi}$ in $\mathscr{S}(\mathbb{R}^d)$, then

$$(\Phi_n,\overline{\Phi}_n)_{q_\tau\otimes q_\sigma} \to (\Phi,\overline{\Phi})_{q_\tau\otimes q_\sigma}, \quad (\Phi_n,S\overline{\Phi}_n)_{q_\tau\otimes q_\sigma} \to (\Phi,S\overline{\Phi})_{q_\tau\otimes q_\sigma}.$$

On the other hand, it is known (cf. Bojdecki and Jakubowski, 1989, proof of Theorem 4.2) that the barrelledness property of $\mathscr{S}(\mathbb{R}^d)$ implies the existence of a continuous Hilbertian seminorm q on $\mathscr{S}(\mathbb{R}^d)$ such that $q_\tau \leqslant q$ for each $\tau \in [0,1]$. Hence, for $\Phi, \overline{\Phi} \in \mathscr{S}(\mathbb{R}^d)\otimes\mathscr{S}(\mathbb{R}^d)$ of the form (2.4),

$$\begin{aligned}
|(\Phi,\overline{\Phi})_{q_\tau\otimes q_\sigma}| &\leqslant \sum_{i,j} q_\tau(\varphi_i)q_\sigma(\psi_i)q_\tau(\overline{\varphi}_j)q_\sigma(\overline{\psi}_j)\\
&\leqslant \sum_i q(\varphi_i)q(\psi_i)\sum_j q(\overline{\varphi}_j)q(\overline{\psi}_j).
\end{aligned}$$

Consequently, by (5.1),

$$|(\Phi,\overline{\Phi})_{q_\tau\otimes q_\sigma}|\leqslant \|\Phi\|_{q(1)}\|\overline{\Phi}\|_{q(1)}$$

for each $\tau,\sigma\in[0,1]$, and by Lemma 5.1 this inequality extends to $\Phi,\overline{\Phi}\in\mathscr{S}(\mathbb{R}^{2d})$. Then, once more by Lemma 5.1 it is seen that if $\Phi_n\to\Phi$, $\overline{\Phi}_n\to\overline{\Phi}$ in $\mathscr{S}(\mathbb{R}^{2d})$, then

$$\sup_n|(\Phi_n,\overline{\Phi}_n)_{q_\tau\otimes q_\sigma}|\leqslant \sup_n\|\Phi_n\|_{q(1)}\sup_n\|\overline{\Phi}_n\|_{q(1)}<\infty,$$

and the same is clearly true for $(\Phi_n,S\overline{\Phi}_n)$.

This proves that condition (i) of Theorem 2.4 is satisfied.

Now, the conditions for existence and non-existence of SILT follow immediately from the previous result and Corollary 5.2. □

Proof of Proposition 3.1.4. Denote by $\|\cdot\|_0$ and $(\cdot,\cdot)_0$, respectively, the norm and the inner product in $L^2(\mathbb{R}^d)$. Replacing A by $(AA^*)^{1/2}$ it can be assumed that A is self-adjoint. Let

$$A\varphi=\sum_k\lambda_k h_k(\varphi,h_k)_0$$

be its spectral decomposition, where $(h_k)_k$ is an orthonormal base in $L^2(\mathbb{R}^d)$ and $\sum_k\lambda_k^2<\infty$. It is clear that $e_k=(1/\lambda_k)h_k$, defined for all k such that $\lambda_k\neq 0$, is an orthonormal base in $\mathscr{S}_q(\mathbb{R}^d)$.

Fix any $f,\overline{f}\in\mathscr{F}$ and $\varepsilon,\delta>0$. We have

$$q(f_\varepsilon(x-\cdot),e_k)=(f_\varepsilon(x-\cdot),A^2e_k)_0=\lambda_k f_\varepsilon * h_k$$

and analogously,

$$q(\varphi\overline{f}_\delta(x-\cdot),e_k)=\lambda_k\overline{f}_\delta*(\varphi h_k)$$

for any $\varphi\in\mathscr{S}(\mathbb{R}^d)$.

By (3.1.1) we have

$$(\Phi^f_{\varepsilon,\varphi},\Phi^{\overline{f}}_{\delta,\varphi})_{q\otimes q}=\sum_k\lambda_k^2(A(\varphi(f_\varepsilon*h_k)),A(\varphi(\overline{f}_\delta*h_k)))_0,$$

$$(\Phi^f_{\varepsilon,\varphi},S\Phi^{\overline{f}}_{\delta,\varphi})_{q\otimes q}=\sum_k\lambda_k^2(A(\varphi(f_\varepsilon*h_k),A(\overline{f}_\delta*(\varphi h_k)))_0.$$

It is well known that

$$\lim_{\varepsilon\to0}f_\varepsilon*h_k=h_k,\quad \lim_{\delta\to0}\overline{f}_\delta*(\varphi h_k)=\varphi h_k\quad\text{in }L^2(\mathbb{R}^d),$$

hence

$$\lim_{\varepsilon,\delta\to0}(A(\varphi(f_\varepsilon*h_k)),A(\varphi(\overline{f}_\delta*h_k)))_0=\|A(\varphi h_k)\|_0^2,$$

$$\lim_{\varepsilon,\delta\to0}(A(\varphi(f_\varepsilon*h_k)),A(\overline{f}_\delta*(\varphi h_k)))_0=\|A(\varphi h_k)\|_0^2\,.$$

On the other hand, denoting by $\|A\|$ the operator norm of A,

$$|(A(\varphi(f_\varepsilon * h_k)), A(\varphi(\overline{f}_\delta * h_k)))_0|$$

$$\leqslant \|A\|^2 \|\varphi(f_\varepsilon * h_k)\|_0 \|\varphi(\overline{f}_\delta * h_k)\|_0$$

$$\leqslant \|A\|^2 \sup_{x\in\mathbb{R}^d} |\varphi(x)|^2 \ \|f_\varepsilon * h_k\|_0 \|\overline{f}_\delta * h_k\|_0 \ \leqslant \ \|A\|^2 \sup_{x\in\mathbb{R}^d} |\varphi(x)|^2,$$

since, by a known estimate of the convolution (e.g., Treves, 1967, Theorem 26.1),

$$\|f_\varepsilon * h_k\|_0 \leqslant \|f_\varepsilon\|_{L^1(\mathbb{R}^d)} \|h_k\|_0 = 1.$$

The expression involving $\overline{f}_\delta * (\varphi h_k)$ is estimated similarly.

Hence, existence of SILT is proved by Proposition 3.1.2 since $\sum_k \lambda_k^2 < \infty$.

To prove continuity observe that in this case formula (5.5) has the form

$$E(\langle L_\varepsilon^f(t_2), \varphi\rangle - \langle L_\varepsilon^f(t_1), \varphi\rangle)^2$$

$$= \int_{[0,1]^4} (1_{[0,t_2]^2}(s,r) - 1_{[0,t_1]^2}(s,r))$$

$$(1_{[0,t_2]^2}(u,v) - 1_{[0,t_1]^2}(u,v))(s\wedge u)(r \wedge v)\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v$$

$$\times((\Phi_{\varepsilon,\varphi}^f, \Phi_{\varepsilon,\varphi}^f)_{q\otimes q} + (\Phi_{\varepsilon,\varphi}^f, S\Phi_{\varepsilon,\varphi}^f)_{q\otimes q}).$$

The expression on the right-hand side is estimated above by

$$(t_2^2 - t_1^2)^2 \sup_{0<\varepsilon<1} ((\Phi_{\varepsilon,\varphi}^f, \Phi_{\varepsilon,\varphi}^f)_{q\otimes q} + (\Phi_{\varepsilon,\varphi}^f, S\Phi_{\varepsilon,\varphi}^f)_{q\otimes q}),$$

so continuity of SILT can be obtained by the same argument as in the proof of Theorem 2.4. □

6.2. *Proofs for Section 3.2*

The SILT existence proofs are based on the following lemma.

Lemma 6.2.1. *Let m, n be non-negative measurable bounded functions on $\mathbb{R}^d$. For $a, a', b, b' > 0$ and $\Phi, \overline{\Phi} \in \mathscr{S}(\mathbb{R}^{2d})$, let*

$$H_{a,a',b,b'}(\Phi, \overline{\Phi}) \tag{6.2.1}$$

$$= \int_{\mathbb{R}^{6d}} m(x)n(y)\Phi(z,w)\overline{\Phi}(z',w')p_a(x,z)p_{a'}(x,z')p_b(y,w)p_{b'}(y,w')$$

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}z'\,\mathrm{d}w\,\mathrm{d}w'.$$

Then

(a) *$H_{a,a',b,b'}$ is a continuous functional on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$, which, moreover, is equicontinuous with respect to a, a', b, b'.*

(b) *For any* $f, \overline{f} \in \mathscr{F}$, $\varphi \in S(\mathbb{R}^d)$,

$$\lim_{\varepsilon,\delta\to 0} H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi}) = \lim_{\varepsilon,\delta\to 0} H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, S\Phi^{\overline{f}}_{\delta,\varphi})$$

$$= \int_{\mathbb{R}^{4d}} m(x)n(y)\varphi(z)\varphi(z')p_a(x,z)p_{a'}(x,z')p_b(y,z)p_{b'}(y,z')\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}z'. \quad (6.2.2)$$

(c) *For any* $f, \overline{f} \in \mathscr{F}$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$,

$$|H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})| \leqslant C(\varphi)(a+a'+b+b')^{-d/\alpha}, \quad (6.2.3)$$

where the constant $C(\varphi)$ *does not depend on* $\varepsilon, \delta, f, \overline{f}, a, a', b, b'$, *and the same inequality holds for* $H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, S\Phi^{\overline{f}}_{\delta,\varphi})$.

Proof. (a) Let M be an upper bound for m and n. We have

$$\begin{aligned}
&|H_{a,a',b,b'}(\Phi,\overline{\Phi})| \\
&\leqslant M^2 \sup_{z,w}|\Phi(z,w)| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\overline{\Phi}(z',w')| \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_a(x,z)p_{a'}(x,z')\ \mathrm{d}x\,\mathrm{d}z\right) \\
&\quad\times \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_b(y,w)p_{b'}(y,w')\ \mathrm{d}y\,\mathrm{d}w\right)\mathrm{d}z'\mathrm{d}w' \\
&= M^2 \sup_{z,w}|\Phi(z,w)| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\overline{\Phi}(z',w')|\mathrm{d}z'\,\mathrm{d}w'.
\end{aligned}$$

Hence the statement follows immediately.

(b) Denote

$$g_{n,b,b'}(w,w') = \int_{\mathbb{R}^d} n(y)p_b(y,w)p_{b'}(y,w')\,\mathrm{d}y.$$

$g_{n,b,b'}$ is continuous and, by the Chapman–Kolmogorov formula,

$$g_{n,b,b'}(w,w') \leqslant M p_{b+b'}(w,w'), \quad (6.2.4)$$

so it is bounded.

We have, for any $f, \overline{f} \in \mathscr{F}$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $\varepsilon, \delta > 0$,

$$\begin{aligned}
&H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi}) \quad (6.2.5)\\
&= \int_{\mathbb{R}^{2d}} \varphi(z)\varphi(z')g_{m,a,a'}(z,z') \int_{\mathbb{R}^{2d}} f_\varepsilon(z-w)\overline{f}_\delta(z'-w')g_{n,b,b'}(w,w')\,\mathrm{d}w\,\mathrm{d}w'\,\mathrm{d}z\,\mathrm{d}z'.
\end{aligned}$$

The inner integral $\int \cdots \mathrm{d}w\,\mathrm{d}w'$ has the form $(f_\varepsilon \otimes \overline{f}_\delta) * g_{n,b,b'}$, so it converges pointwise to $g_{n,b,b'}(z,z')$ as $\varepsilon, \delta \to 0$. Existence of $\lim_{\varepsilon,\delta\to 0} H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})$ and formula (6.2.2) now follow by the dominated convergence theorem, since $\int f_\varepsilon(z-w)\overline{f}_\delta(z'-w')\,\mathrm{d}w\,\mathrm{d}w' = 1$ and $\int |\varphi(z)\varphi(z')|\mathrm{d}z\,\mathrm{d}z' < \infty$.

To prove the existence of $\lim_{\varepsilon,\delta\to 0} H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, S\Phi^{\overline{f}}_{\delta,\varphi})$, first observe that the function $(w,w') \mapsto \varphi(w')g_{n,b,b'}(w,w')$ is square-integrable on $\mathbb{R}^{2d}$. Indeed, by (6.2.4),

$$\int_{\mathbb{R}^{2d}} \varphi^2(w')g^2_{n,b,b'}(w,w')\mathrm{d}w\,\mathrm{d}w' \leqslant M^2 \int_{\mathbb{R}^d} \varphi^2(w') \int_{\mathbb{R}^d} (p_{b+b'}(w,w'))^2 \mathrm{d}w\,\mathrm{d}w'$$
$$\leqslant M^2 C(\alpha, b+b') \int_{\mathbb{R}^d} \varphi^2(w')\,\mathrm{d}w' < \infty.$$

We have

$$H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, S\Phi^{\overline{f}}_{\delta,\varphi})$$
$$= \int_{\mathbb{R}^{2d}} \varphi(z)g_{m,a,a'}(z,z') \int_{\mathbb{R}^{2d}} f_\varepsilon(z-w)\overline{f}_\delta(z'-w')\varphi(w')g_{n,b,b'}(w,w')\,\mathrm{d}w\,\mathrm{d}w'\,\mathrm{d}z\,\mathrm{d}z'.$$

The inner integral has the form $(f_\varepsilon \otimes \overline{f}_\delta) * (\varphi g_{n,b,b'})$, so, by the previous remark it converges (to $\varphi g_{n,b,b'}$) in $L^2(\mathbb{R}^{2d})$ as $\varepsilon,\delta \to 0$. Since, again by the previous remark, $\varphi g_{m,a,a'} \in L^2(\mathbb{R}^{2d})$, formula (6.2.2) is proved.

(c) To obtain (6.2.3) note that by (6.2.4) and (6.2.5),

$$|H_{a,a',b,b'}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})|$$
$$\leqslant M^2 \sup_{z'} |\varphi(z')| \int_{\mathbb{R}^{2d}} |\varphi(z)|\,p_{a+a'}(z,z')$$
$$\times \int_{\mathbb{R}^{2d}} f_\varepsilon(z-w)\overline{f}_\delta(z'-w')p_{b+b'}(w,w')\,\mathrm{d}w\,\mathrm{d}w'\,\mathrm{d}z\,\mathrm{d}z'.$$

The right-hand side can be rewritten by a change of variables, by using the fact that $p_{b+b'}(w,w')$ depends on $w-w'$ only, and by changing the order of integration, in the following way:

$$M^2 \sup_{z'} |\varphi(z')| \int_{\mathbb{R}^d} |\varphi(z)|$$
$$\times \int_{\mathbb{R}^{2d}} f_\varepsilon(w)\overline{f}_\delta(w') \int_{\mathbb{R}^d} p_{a+a'}(z,z')p_{b+b'}(z', z-w+w')\mathrm{d}z'\,\mathrm{d}w\,\mathrm{d}w'\,\mathrm{d}z.$$

The inner integral ($\int \cdots \mathrm{d}z'$), by the Chapman–Kolmogorov formula, is equal to

$$p_{a+a'+b+b'}(z, z-w+w') = p_{a+a'+b+b'}(w,w') \leqslant C(\alpha)(a+a'+b+b')^{-d/\alpha},$$

where we have used the self-similarity of the stable density. Eq. (6.2.3) is thus proved since $\int\int f_\varepsilon(w)\overline{f}_\delta(w')\mathrm{d}w\,\mathrm{d}w' = 1$.

The inequality involving $S\Phi^{\overline{f}}_{\delta,\varphi}$ is derived in the same way. □

Remark 6.2.2. (a) Lemma 6.2.1, with obvious modifications in the formulation and in the proof, holds also if a and/or b are equal to 0.

(b) Lemma 6.2.1 also remains valid if the functions m, n depend on additional parameters which may be related to a, a', b, b', provided that the bound M does not depend on them.

For the non-existence proofs the following lemma is useful.

Lemma 6.2.3. *Let m be a non-negative measurable function on $\mathbb{R}^d$, and assume that $\inf_{y\in\Gamma_m} m(y) > c_m$ for some $c_m > 0$, and $\Gamma_m \in \mathscr{B}(\mathbb{R}^d)$, $\lambda(\Gamma_m) > 0$. Then, for each $T > 0$ and $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $\varphi > 0$, there exists a positive constant $C(\varphi)$ such that*

$$\int_{\mathbb{R}^{2d}} m(y)p_a(x,y)p_b(x,y)\varphi(x)\varphi(y)\,\mathrm{d}x\,\mathrm{d}y \geqslant c_m C(\varphi)(a+b)^{-d/\alpha} \tag{6.2.6}$$

for all $a, b \in [0, T]$.

Proof. By the self-similarity of the stable density we have $p_{a+b}(0,0) = C(\alpha)(a+b)^{-d/\alpha}$, so the left-hand side of (6.2.6) can be written as

$$\begin{aligned} &C(\alpha)(a+b)^{-d/\alpha}\int_{\mathbb{R}^{2d}} m(y)\varphi(x)\varphi(y)\frac{p_a(0,y-x)p_b(0,y-x)}{p_{a+b}(0,0)}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = C(\alpha)(a+b)^{-d/\alpha}\int_{\mathbb{R}^d} \psi(x)\frac{p_a(0,x)p_b(x,0)}{p_{a+b}(0,0)}\,\mathrm{d}x, \end{aligned}$$

where $\psi(x) = \int m(y)\varphi(y)\varphi(y-x)\,\mathrm{d}y$. By the assumptions on m we have $\psi(x) \geqslant c_m\psi_1(x)$, where $\psi_1(x) = \int_{\Gamma_m} \varphi(y)\varphi(y-x)\,\mathrm{d}y$.

Denote by $(Z_t^{a+b})_{t\in[0,a+b]}$ the α-stable bridge between the points (in time-space) $(0,0)$ and $(a+b,0)$. Then $p_a(0,\cdot)p_b(\cdot,0)/p_{a+b}(0,0)$ is the density of Z_a^{a+b} (see, e.g., Jamison, 1974), so the left-hand side of (6.2.6) is bounded below by

$$C(\alpha)c_m(a+b)^{-d/\alpha}E(\psi_1(Z_a^{a+b})).$$

Hence, to complete the proof it suffices to show that

$$\inf_{a,b\in(0,T]} E(\psi_1(Z_a^{a+b})) > 0. \tag{6.2.7}$$

Note that ψ_1 is bounded, continuous and strictly positive. Suppose that (6.2.7) does not hold. Then there exist sequences $(a_n)_n, (b_n)_n$ which can be assumed to converge, $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, such that

$$\lim_{n\to\infty} E(\psi_1(Z_{a_n}^{a_n+b_n})) = 0.$$

If $a > 0, b > 0$, then we obtain a contradiction immediately since the function $t \mapsto p_t(x,y)$ is continuous on $\mathbb{R}_+\backslash\{0\}$, so

$$\lim_{n\to\infty} E(\psi_1(Z_{a_n}^{a_n+b_n})) = E(\psi_1(Z_a^{a+b})) > 0.$$

If $a = 0$, we want to prove that

$$\lim_{n\to\infty} E(\psi_1(Z_{a_n}^{a_n+b_n})) = \psi(0), \tag{6.2.8}$$

which, clearly, will give the desired contradiction. Eq. (6.2.8) will be proved if we show that for each $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|Z_{a_n}^{a_n+b_n}| > \varepsilon) = 0. \tag{6.2.9}$$

To this end, first observe that by standard properties of the stable density, if $\|x\| > \varepsilon$ and $a', b' \in (0, T]$, then

$$\frac{p_{b'}(x,0)}{p_{a'+b'}(0,0)} \leqslant C(\alpha)^{-1}(a'+b')^{d/\alpha} \sup_{\|x\|=\varepsilon} p_{b'}(x,0)$$

$$\leqslant C(\alpha)^{-1}(2T)^{d/\alpha} \sup_{b'\in(0,T]} \sup_{\|x\|=\varepsilon} p_{b'}(x,0) < \infty.$$

Let

$$M = \sup_{a',b'\in(0,T]} \sup_{\|x\|>\varepsilon} \frac{p_{b'}(x,0)}{p_{a'+b'}(0,0)}.$$

We then have

$$\int_{\|x\|>\varepsilon} p_{a_n}(0,x)\left(M - \frac{p_{b_n}(x,0)}{p_{a_n+b_n}(0,0)}\right) \mathrm{d}x \geqslant 0$$

for $n = 1, 2, \ldots$, hence

$$P(|Z_{a_n}^{a_n+b_n}| > \varepsilon) = \int_{\|x\|>\varepsilon} \frac{p_{a_n}(0,x)p_{b_n}(x,0)}{p_{a_n+b_n}(0,0)} \mathrm{d}x \leqslant M \int_{\|x\|>\varepsilon} p_{a_n}(0,x)\,\mathrm{d}x,$$

and this implies (6.2.9) since the symmetric α-stable process is right-continuous and equal to 0 at $t = 0$.

Finally, if $b = 0$, the argument is identical by symmetry. □

Remark 6.2.4. Suppose that m depends on some additional parameters which may be related to a, b, and $\inf_{y\in\Gamma_m} m(y) > c_m$ for these parameters belonging to some set Z. Let $A = \{(a,b) :$ the parameters are in $Z\}$. If $T > 0$ is such that $[0,T] \times [0,T] \subset A$, then (6.2.6) still holds for $a, b \in [0, T]$. This sort of dependence will become clear in the proof below.

Proof of Proposition 3.2.1. (a) If the covariance functional of X has the form (3.2.2), then it is easy to see from (2.5) that

$$\begin{aligned} &J_{s,r,u,v}(\Phi,\overline{\Phi}) \\ &= \int_{\mathbb{R}^{4d}} m(s\wedge u,x)m(r\wedge v,y)\Phi(x,y)\overline{\Phi}(z',w')p_{|s-u|}(x,z')p_{|r-v|}(y,w')\,\mathrm{d}z'\,\mathrm{d}w'\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^{4d}} m(s\wedge v,x)m(r\wedge u,y)\Phi(x,y)S\overline{\Phi}(z',w')p_{|s-v|}(x,z')p_{|r-u|}(y,w')\mathrm{d}z'\,\mathrm{d}w'\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Lemma 6.2.1 and Remark 6.2.2 imply that conditions (i) and (ii) of Theorem 2.4 are satisfied and (2.6) is fulfilled with

$$G_\varphi(s,r,u,v) = C(\varphi)((|s-u| + |r-v|)^{-d/\alpha} + (|s-v| + |r-u|)^{-d/\alpha})$$

The existence and continuity part now follows from Theorem 2.4 and Lemma 2.6 (a), since $d < 2\alpha$.

Now, let $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $\varphi > 0$, $f \in \mathscr{F}$. By Fatou's lemma, Eq. (6.2.2) and Remark 6.2.2(b) we have

$$\liminf_{\varepsilon \to 0} \int_{[0,t]^4} J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^f_{\varepsilon,\varphi})\, \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v$$
$$\geqslant \int_{[0,t]^4} \left[\int_{\mathbb{R}^{2d}} m(s \wedge u, x) m(r \wedge v, x) \varphi(x) \varphi(z') p_{|s-u|}(x,z') p_{|r-v|}(x,z')\, \mathrm{d}x\, \mathrm{d}z' \right.$$
$$\left. + \int_{\mathbb{R}^{2d}} m(s \wedge v, x) m(r \wedge u, x) \varphi(x) \varphi(z') p_{|s-v|}(x,z') p_{|r-u|}(x,z') \mathrm{d}x \mathrm{d}z' \right] \mathrm{d}s\, \mathrm{d}v\, \mathrm{d}u\, \mathrm{d}v$$
$$\geqslant c_m^2 C(\varphi) \int_{[0,t]^4} [(|s-u| - |r-v|)^{-\frac{d}{\alpha}} + (|s-v| + |r-u|)^{-d/\alpha}] \mathrm{d}s\, \mathrm{d}v\, \mathrm{d}u\, \mathrm{d}v,$$

where we have used assumption (3.2.1), Lemma 6.2.3 and Remark 6.2.4 for t sufficiently small. By Lemma 2.6 the right-hand side is infinite if $d \geqslant 2\alpha$, and the proof is complete.

(b) If the covariance of X has the form (3.2.3), then

$$J_{s,r,u,v}(\Phi, \overline{\Phi}) = \int_{\mathbb{R}^{6d}} m(s \wedge u, x) m(r \wedge v, y) \Phi(z,w) \overline{\Phi}(z', w')$$
$$\times p_s(x,z) p_u(x,z') p_r(y,w) p_v(y,w')\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}z'\, \mathrm{d}w'$$
$$+ \int_{\mathbb{R}^{6d}} m(s \wedge v, x) m(r \wedge u, y) \Phi(z,w) S\overline{\Phi}(z', w')$$
$$\times p_s(x,z) p_v(x,z') p_r(y,w) p_u(y,w')\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}z'\, \mathrm{d}w',$$

so, if $d < 4\alpha$, it is seen that, by Lemma 6.2.1 and Remark 6.2.2(b), the assumptions of Theorem 2.4 are satisfied with G_φ given in Lemma 2.6(b) and we have existence and continuity.

To prove non-existence for $d \geqslant 4\alpha$ we argue as in the first part of the proof, using, among others, Lemma 6.2.3 and Remark 6.2.4.

If the covariance of X has the form (3.2.4), then

$$J_{s,r,u,v}(\Phi, \overline{\Phi})$$
$$= \int_0^{s \wedge u} \int_0^{r \wedge v} \int_{\mathbb{R}^{6d}} m(\tau, x) m(\sigma, y) \Phi(z,w) \overline{\Phi}(z', w')$$
$$\times\ p_{s-\tau}(x,z) p_{u-\tau}(x,z') p_{r-\sigma}(y,w) p_{v-\sigma}(y,w') \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}z'\, \mathrm{d}w' \mathrm{d}\sigma \mathrm{d}\tau$$
$$+ \int_0^{s \wedge v} \int_0^{r \wedge u} \int_{\mathbb{R}^{6d}} m(\tau, x) m(\sigma, y) \Phi(z,w) S\overline{\Phi}(z', w')$$
$$\times\ p_{s-\tau}(x,z) p_{v-\tau}(x,z') p_{r-\sigma}(y,w) p_{u-\sigma}(y,w') \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}z' \mathrm{d}w' \mathrm{d}\sigma \mathrm{d}\tau,$$

and once more we apply Lemmas 6.2.1 and 6.2.3 to prove the result (in this case the function G_φ has the form given in Lemma 2.6 (c)). □

Proof of Proposition 3.2.3. In formula (2.5), besides the terms involving the products $K_1(\cdot)K_1(\cdot), K_2(\cdot)K_2(\cdot)$, there will appear summands of mixed type, i.e., $K_1(\cdot)K_2(\cdot)$. However, Lemma 6.2.1 permits to treat these terms as well and we can argue as in the preceding proofs for existence and continuity of SILT.

The non-existence part is obvious. □

6.3. Proofs for Section 3.4

The proof for the Poisson system of α-stable bridges is very similar to that of Proposition 3.2.1(a), with some differences in the computations. We do not include it.

Proof of Proposition 3.4.1. Recall that $d > \alpha$ is necessary for existence of the measure R_∞.

We will show that SILT exists for $2\alpha < d < 6\alpha$ by verifying the conditions of Theorem 2.4.

Formula (2.5) takes the form

$$J_{s,r,u,v}(\Phi,\overline{\Phi}) = J^{(1)}_{s,r,u,v}(\Phi,\overline{\Phi}) + J^{(2)}_{s,r,u,v}(\Phi,\overline{\Phi}), \quad \Phi,\overline{\Phi} \in \mathscr{S}(\mathbb{R}^{2d}),$$

where

$$J^{(1)}_{s,r,u,v}(\Phi,\overline{\Phi}) = \int_{\mathbb{R}^{8d}} \frac{\Phi(x_2,x_3)\overline{\Phi}(y_2,y_3)}{(\|x-y\|\|x_1-y_1\|)^{d-\alpha}} \times p_s(x,x_2)p_r(x_1,x_3)p_u(y,y_2)p_v(y_1,y_3)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_3\,\mathrm{d}y_3, \tag{6.3.1}$$

and $J^{(2)}_{s,r,u,v}(\Phi,\overline{\Phi})$ is a similar term. (We are abusing notation by writing J when we still have to verify that it is well-defined on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$, but this will be a consequence of the following calculations.) Since $J^{(2)}$ yields the same results as $J^{(1)}$ (with a minor change in the computations, which will be mentioned below), we continue calculations only for $J^{(1)}$.

We will need the following result from Iscoe (1986), which will be used several times.

Lemma 6.3.1. *If ψ is a bounded measurable function on $\mathbb{R}^d$ which is $\mathrm{O}(\|x\|^{-p})$ as $\|x\| \to \infty$ for some $p > d$, and φ is defined on $\mathbb{R}^d$ by*

$$\varphi(x) = \int_{\mathbb{R}^d} \frac{\psi(y)}{\|x-y\|^{d-\alpha}}\mathrm{d}y \quad (d > \alpha),$$

then

$$\sup_x (1+\|x\|^{d-\alpha})|\varphi(x)| \leqslant C \sup_x (1+\|x\|^p)|\psi(x)| \tag{6.3.2}$$

for some positive constant C.

We will also use the fact that

$$p_1(x) = \mathrm{O}(\|x\|^{-(d+\alpha)}) \quad \text{as} \quad \|x\| \to \infty, \tag{6.3.3}$$

(see Gorostiza and Wakolbinger, 1991, (5.3)).

Making the change of variables $x = y + z$, $x_1 = y_1 + z_1$ in (6.3.1) and using the Chapman–Kolmogorov formula we obtain

$$J^{(1)}_{s,r,u,v}(\Phi,\overline{\Phi}) = \int_{\mathbb{R}^{4d}} \Phi(x_2,x_3)\overline{\Phi}(y_2,y_3)\left(\int_{\mathbb{R}^d} \frac{p_{s+u}(x_2-y_2-z)}{\|z\|^{d-\alpha}}\mathrm{d}z\right) \times \left(\int_{\mathbb{R}^d} \frac{p_{r+v}(x_3-y_3-z_1)}{\|z_1\|^{d-\alpha}}\mathrm{d}z_1\right)\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_3\,\mathrm{d}y_3. \tag{6.3.4}$$

Using Lemma 6.3.1 we obtain

$$\int_{\mathbb{R}^d} \frac{p_t(x-z)}{\|z\|^{d-\alpha}}\,\mathrm{d}z = \int_{\mathbb{R}^d} \frac{p_t(z)}{\|x-z\|^{d-\alpha}}\,\mathrm{d}z \leqslant C_1\,\|x\|^{-(d-\alpha)}$$

for all s, $u > 0$. Indeed, using the self-similarity of $p_t(x)$ we have

$$\int \frac{p_t(z)}{\|x-z\|^{d-\alpha}}\mathrm{d}z = t^{-d/\alpha}\int \frac{p_1(t^{-1/\alpha}z)}{\|x-z\|^{d-\alpha}}\mathrm{d}z = \int \frac{p_1(z')}{\|x-t^{1/\alpha}z'\|^{d-\alpha}}\mathrm{d}z' = t^{-d/\alpha+1}\int \frac{p_1(z')}{\|t^{-1/\alpha}x-z'\|^{d-\alpha}}\mathrm{d}z',$$

and the result follows from (6.3.3) and (6.3.2) with $p = d+\alpha$. (This can also be proved using the potential of $p_t(z)$.)

Hence, from (6.3.4) we have, for all s, r, u, $v > 0$,

$$|J^{(1)}_{s,r,u,v}(\Phi,\overline{\Phi})| \leqslant C_2\int_{\mathbb{R}^{4d}} \frac{|\Phi(x_2,x_3)\overline{\Phi}(y_2,y_3)|}{(\|x_2-y_2\|\|x_3-y_3\|)^{d-\alpha}}\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_3\,\mathrm{d}y_3 = C_2\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^d}\frac{|\Phi(x_2,x_3)|}{\|x_2-y_2\|^{d-\alpha}}\mathrm{d}x_2\right) \left(\int_{\mathbb{R}^d}\frac{|\overline{\Phi}(y_2,y_3)|}{\|x_3-y_3\|^{d-\alpha}}\mathrm{d}y_3\right)\mathrm{d}y_2\,\mathrm{d}x_3.$$

Since $\Phi \in \mathscr{S}(\mathbb{R}^{2d})$, by Lemma 6.3.1 we have

$$\int_{\mathbb{R}^d}\frac{|\Phi(x_2,x_3)|}{\|x_2-y_2\|^{d-\alpha}}dx_2 \leqslant \frac{C_3}{\|y_2\|^{d-\alpha}}\sup_x(1+\|x\|^p)|\Phi(x,x_3)|,$$

for any $p > d$, and similarly for the integral in y_3. Hence,

$$\begin{aligned}
&|J^{(1)}_{s,r,u,v}(\Phi,\overline{\Phi})| \\
&\leqslant C_4\int_{\mathbb{R}^{2d}}\sup_x(1+\|x\|^p)|\Phi(x,x_3)|\sup_y(1+\|y\|^p)|\overline{\Phi}(y_2,y)|\frac{\mathrm{d}y_2\,\mathrm{d}x_3}{(\|y_2\|\|x_3\|)^{d-\alpha}} \\
&\leqslant C_4\sup_{x,x'}(1+\|x\|^p)(1+\|x'\|^k)|\Phi(x,x')|\sup_{y,y'}(1+\|y\|^p)(1+\|y'\|^k)|\overline{\Phi}(y',y)| \\
&\quad\times\left(\int_{\mathbb{R}^d}\frac{\mathrm{d}z}{\|z\|^{d-\alpha}(1+\|z\|^k)}\right)^2,
\end{aligned} \tag{6.3.5}$$

with k large enough so that the integral in z exists (integrability at the origin holds because $\alpha > 0$). It follows from (6.3.5) that $J^{(1)}_{s,r,u,v}(\Phi,\overline{\Phi})$ is well-defined and continuous on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$ and moreover, since the bound is independent of s,r,u,v, condition (i) of Theorem 2.4 is satisfied.

We turn to condition (ii). Putting $\Phi = \Phi^{f}_{\varepsilon,\varphi}$ and $\overline{\Phi} = \Phi^{\overline{f}}_{\delta,\varphi}$ defined by (2.2) into (6.3.4), using the fact that $\|z\|^{-(d-\alpha)}$ is the potential of $p_t(z)$ (e.g., Blumenthal and Getoor, 1968) and the Chapman–Kolmogorov formula, we obtain

$$J^{(1)}_{s,r,u,v}(\Phi^{f}_{\delta,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi}) = C_5 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{4d}} \varphi(x) f_\varepsilon(x-z)\varphi(y)\overline{f}_\delta(y-w) \times p_{s+u+\tau}(x-y)p_{r+v+\sigma}(z-w)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}\tau\mathrm{d}\sigma. \tag{6.3.6}$$

Clearly,

$$\lim_{\varepsilon,\delta\to 0} \int_{\mathbb{R}^{2d}} f_\varepsilon(x-z)\overline{f}_\delta(y-w)p_{r+v+\sigma}(z-w)\mathrm{d}z\,\mathrm{d}w = p_{r+v+\sigma}(x-y),$$

so we must justify the interchange of $\lim_{\varepsilon,\delta}$ and integration on x, y, τ, σ in (6.3.6). By the self-similarity of $p_t(x)$ we have

$$p_{s+u+\tau}(x-y)p_{r+v+\sigma}(z-w) \leqslant C_6(s+u+\tau)^{-d/\alpha}(r+v+\sigma)^{-d/\alpha}.$$

Hence

$$\left|\varphi(x)\varphi(y)p_{s+u+\tau}(x-y)\int_{\mathbb{R}^{2d}} f_\varepsilon(x-z)\overline{f}_\delta(y-w)p_{r+v+\sigma}(z-w)\mathrm{d}z\,\mathrm{d}w\right| \leqslant C_6(s+u+\tau)^{-d/\alpha}(r+v+\sigma)^{-d/\alpha}|\varphi(x)||\varphi(y)|, \tag{6.3.7}$$

which is integrable in x, y, τ, σ for each $s, r, u, v > 0$, since $d > \alpha$. (Note: In the calculation for $J^{(2)}$, in formula (6.3.7), $|\varphi(y)|$ is replaced by $\int \overline{f}_\delta(y-w)|\varphi(w)|\mathrm{d}w$, and a little extra work is necessary in order to obtain a dominating function). Therefore, the interchange is allowed by the Lebesgue dominated convergence theorem, and we have

$$\lim_{\varepsilon,\delta\to 0} J^{(1)}_{s,r,u,v}(\Phi^{f}_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi}) = C_5 \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{2d}} \varphi(x)\varphi(y)p_{s+u+\tau}(x-y)p_{r+v+\sigma}(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\tau\,\mathrm{d}\sigma \tag{6.3.8}$$

for each $f, \overline{f} \in \mathscr{F}$, $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $s, r, u, v \in (0,1]$, and we note that the limit does not depend on $f, \overline{f}$. So condition (ii) is satisfied.

We now seek the function G_φ for condition (iii). Putting $\Phi = \Phi^{f}_{\varepsilon,\varphi}$, $\overline{\Phi} = \Phi^{\overline{f}}_{\delta,\varphi}$ into (6.3.1), taking a bound for $|\varphi(y_2)|$, and changing variables of integration so as to make convenient use of commutativity of convolution, we obtain

$$\begin{aligned}
&|J^{(1)}_{s,r,u,v}(\Phi^{f}_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})| \\
&\leqslant C(\varphi)\int_{\mathbb{R}^{8d}} \frac{|\varphi(x_2)|}{(\|z\|\|z_1\|)^{d-\alpha}} p_r(x_2,x_3)p_v(x_3,y_1+z_1)p_u(y_1+z_1,y_3+z_1) \\
&\quad\times p_s(y_3+z_1,y_2+z_1)f_\varepsilon(y_2-y)\overline{f}_\delta(y-x_2+z)\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}z_1\,\mathrm{d}y_1\,\mathrm{d}x_2\,\mathrm{d}y_2\,\mathrm{d}x_3\,\mathrm{d}y_3 \\
&= C(\varphi)\int_{\mathbb{R}^{4d}} \frac{|\varphi(x_2)|}{(\|z\|\|z_1\|)^{d-\alpha}} p_{s+r+u+v}(x_2,y_2+z_1)f_\varepsilon * \overline{f}_\delta(y_2-x_2+z)\mathrm{d}z\,\mathrm{d}z_1\,\mathrm{d}x_2\,\mathrm{d}y_2,
\end{aligned}$$

where we have used the Chapman–Kolmogorov formula in the last step.

Since f has bounded support, by two successive applications of Lemma 6.3.1 we obtain

$$\int_{\mathbb{R}^d} \frac{f_\varepsilon * \overline{f}_\delta(y+z)}{\| z \|^{d-\alpha}} \mathrm{d}z \leqslant \frac{C_7}{\| y \|^{d-\alpha}}$$

for all $\varepsilon, \delta \in (0,1)$, where the constant C_7 is independent of ε, δ. Indeed,

$$\begin{aligned} \int \frac{f_\varepsilon * \overline{f}_\delta(y+z)}{\| z \|^{d-\alpha}} dz &= \int\int \frac{f_\varepsilon(y+z-x)\overline{f}_\delta(x)}{\| z \|^{d-\alpha}} \mathrm{d}x\,\mathrm{d}z \\ &= \varepsilon^{-d}\delta^{-d} \int \frac{1}{\| z \|^{d-\alpha}} f\left(\frac{y+z-x}{\varepsilon}\right) \overline{f}\left(\frac{x}{\delta}\right) \mathrm{d}z\mathrm{d}x \\ &= \int \frac{f(z')\overline{f}(x')}{\| \delta x' + \varepsilon z' - y \|^{d-\alpha}} f(z')\overline{f}(x')\,\mathrm{d}z'\,\mathrm{d}x' \\ &= \frac{1}{\delta^{d-\alpha}} \int f(z') \int \frac{\overline{f}(x')}{\| x' + (\varepsilon/\delta)z' - y/\delta \|^{d-\alpha}} \mathrm{d}x'\mathrm{d}z'. \end{aligned}$$

Due to (6.3.2), the inner integrand is bounded above by

$$C \frac{\delta^{d-\alpha}}{\| \varepsilon z' - y \|^{d-\alpha}} \sup_x (1 + \| x \|^p)\overline{f}(x),$$

with any $p > d$. Hence,

$$\int \frac{f_\varepsilon * \overline{f}_\delta(y+z)}{\| z \|^{d-\alpha}} \mathrm{d}z \leqslant C(\overline{f}) \frac{1}{\varepsilon^{d-\alpha}} \int \frac{f(z)}{\| z - y/\varepsilon \|^{d-\alpha}} \mathrm{d}z,$$

and the result follows once again from (6.3.2).

Hence,

$$|J^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})| \leqslant C_1(\varphi) \int_{\mathbb{R}^{3d}} \frac{|\varphi(x)|}{(\| z \| \| x - y \|)^{d-\alpha}} p_{s+r+u+v}(z, x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

and using again the potential of $p_t(z)$, the Chapman–Kolmogorov formula and the self-similarity of $p_t(x)$, we obtain

$$\begin{aligned} |J^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})| &\leqslant C_2(\varphi) \int_{\mathbb{R}^d} |\varphi(x)|\mathrm{d}x \int_0^\infty \int_0^\infty p_{s+r+u+v+\tau+\sigma}(0,0)\,\mathrm{d}\tau\,\mathrm{d}\sigma \\ &= C_3(\varphi) \int_0^\infty \int_0^\infty (s+r+u+v+\tau+\sigma)^{-d/\alpha}\mathrm{d}\tau\,\mathrm{d}\sigma. \end{aligned} \quad (6.3.9)$$

The right-hand side of (6.3.9) is the function $G_\varphi(s,r,u,v)$ (it works for $J^{(2)}$ as well). G_φ is finite because $d > 2\alpha$ implies integrability. Carrying out the integration we obtain

$$G_\varphi(s,r,u,v) \leqslant C_4(\varphi)(s+r+u+v)^{2-d/\alpha}. \quad (6.3.10)$$

Since $0 < d/\alpha - 2 < 4$, then (2.7) holds due to (6.3.10) and Lemma 2.6(b).

Thus, we have shown by Theorem 2.4 that X_1 has SILT for $2\alpha < d < 6\alpha$. Moreover, condition (iv) of Theorem 2.4 also holds, again by (6.3.10) and Lemma 2.6(b), and therefore the SILT process is continuous.

Now we prove that X_1 does not have SILT for $d \leqslant 2\alpha$ and $d \geqslant 6\alpha$. From (6.3.8) and Lemma 6.2.3 we have, for $\varphi > 0$,

$$\lim_{\varepsilon,\delta\to 0} J^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi}) \geqslant C_5(\varphi) \int_0^\infty \int_0^\infty (s+r+u+v+\tau+\sigma)^{-d/\alpha} \mathrm{d}\tau\, \mathrm{d}\sigma.$$

It follows that

$$\int_{[0,t]^4} \lim_{\varepsilon,\delta\to 0} J^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^{\overline{f}}_{\delta,\varphi})\, \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v = \infty,$$

if $d \leqslant 2\alpha$ or $d \geqslant 6\alpha$. Therefore, by Fatou's lemma and Corollary 2.5, X_1 does not have SILT for these dimensions. □

Proof of Proposition 3.4.3. Let $J^X_{s,r,u,v}(\Phi, \overline{\Phi})$ denote the right-hand side of (2.5) and $J^Y_{s,r,u,v}(\Phi, \overline{\Phi})$ the same expression for Y. Then

$$J^Y_{s,r,u,v}(\Phi, \overline{\Phi}) = \int_0^s \int_0^r \int_0^u \int_0^v J^X_{s',r',u',v'}(\Phi, \overline{\Phi}) \mathrm{d}v'\, \mathrm{d}u'\, \mathrm{d}r'\, \mathrm{d}s'.$$

It follows from condition (i) (with $|J^X|$) that $J^Y_{s,r,u,v}$ is well-defined on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$.

By Fubini's theorem,

$$\begin{aligned}
&\left| \int_{[0,t]^4} J^Y_{s,r,u,v}(\Phi, \overline{\Phi}) \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v \right| \\
&= \left| \int_{[0,t]^4} (t-s)(t-r)(t-u)(t-v) J^X_{s,r,u,v}(\Phi, \overline{\Phi}) \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v \right| \\
&\leqslant \int_{[0,t]^4} |J^X_{s,r,u,v}(\Phi, \overline{\Phi})| \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v,
\end{aligned}$$

which by condition (i) for $|J^X|$ implies that (i) also holds for J^Y.

Condition (ii) for J^Y follows from conditions (ii) and (iii) for J^X and the dominated convergence theorem.

Let G^X_φ denote the function in condition (iii) for J^X, let

$$G^Y_\varphi(s,r,u,v) = \int_0^s \int_0^r \int_0^u \int_0^v G^X_\varphi(s',r',u',v') \mathrm{d}v'\, \mathrm{d}u'\, \mathrm{d}r'\, \mathrm{d}s',$$

and note that, by Fubini's theorem as above,

$$\int_{[0,1]^4} G^Y_\varphi(s,r,u,v)\, \mathrm{d}s \mathrm{d}r \mathrm{d}u \mathrm{d}v \leqslant \int_{[0,1]^4} G^X_\varphi(s,r,u,v) \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v.$$

Hence condition (iii) holds for J^Y.

Therefore by Theorem 2.4 SILT exists for Y.

Moreover, condition (iv) holds for Y since

$$\begin{aligned}
&\int_{[0,1]^4} (1_{[0,t_2]^2}(s,r) - 1_{[0,t_1]^2}(s,r))(1_{[0,t_2]^2}(u,v) - 1_{[0,t_1]^2}(u,v)) G^Y_\varphi(s,r,u,v) \mathrm{d}s\, \mathrm{d}r\, \mathrm{d}u\, \mathrm{d}v \\
&\quad \leqslant C(\varphi)(t_2^2 - t_1^2)^2 \leqslant 4C(\varphi)(t_2 - t_1)^2,
\end{aligned}$$

where

$$C(\varphi) = \int_{[0,1]^4} G_\varphi^X(s,r,u,v)\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}v.$$

Hence, the SILT process of Y is continuous. □

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